BIRZEIT UNIVERSITY

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Mathematics Program

LIE SYMMETRY METHODS FOR SOLVING DIFFRENTIAL AND DIFFERENCE EQUATIONS.

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الإهداء

بداية أشكر الله على كل نعمة التي فضّل علي بها، وعلى فضله وكرمه اللذين أوصلاني لهذا اليوم.

إلى نعمتي الكبيرة (أبي وأمي) اللذين برضاهم ومحبتهما وصلت لهذه المرحلة.

إلى عائلتي (أخوتي وزوجاتهم وأبنائهم وأخواتي وزوج أختي وابنتهم).

إلى أصدقائي الذين أحبهم ولن أنساهم ابداً.

إلى زملائي الذين مرّوا في حياتي الدراسية منذ بدايتها.

والشكر موصولٌ إلى مشرفي الدكتور مروان العقيلي.

وإلى لجنة النقاش الدكتور علاء الدين عليان والدكتور علاء تلاحمة.

DECLARATION

I certify that this Thesis, submitted for the degree of Master of Mathematics to the Department of Mathematics at Birzeit University, is of my own research except where otherwise acknowledged, and that this thesis (or any part of it) has not been submitted for a higher degree to any other university or institution.

Isra Abo Baker Nazzal	Signature
July, 2019	....

ABSTRACT

We review the symmetry method that can be used to solve differential and difference equations. We use the symmetry method to solve the sixth order difference equation.

$$u_{n+6} = \frac{u_n}{A_n + B_n u_n u_{n+3}}$$

where the initial values $u_0, u_1, \cdots, u_5$ are arbitrary nonzero real numbers and the eighth order difference equation

$$u_{n+8} = \frac{u_n}{A_n + B_n u_n u_{n+2} u_{n+4} u_{n+6}}$$

where the initial values $u_0, u_1, \cdots, u_7$ are arbitrary nonzero real numbers by determining Lie groups of symmetries.

Keywords: Differential equations, Difference equations, Lie groups, Symmetry method.

الملخص

الهدف من الرسالة هو دراسة طريقة التماثل لحل معادلات الفرق والمعادلات التفاضلية. وسوف نقوم باستعراض هذه الطريقة لحل هذه المعادلات ونستخدمها في حل المعادلة التالية من الدرجه السادسة

$$u_{n+6} = \frac{u_n}{A_n + B_n u_n u_{n+3}}$$

والمعادلة من الدرجة الثامنة

$$u_{n+8} = \frac{u_n}{A_n + B_n u_n u_{n+2} u_{n+4} u_{n+6}}$$

حيث القيم الأولية هي أرقام اختيارية حقيقية غير صفرية من خلال تحديد مجموعات التماثل.

الكلمات المفتاحية: المعادلات الفرق، المعادلات التفاضلية، طريقة التماثل، مجموعات التماثل.

CONTENTS

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1. LITERATURE REVIEW

Nonlinear differential and difference equations have important applications in different fields of science and technology. Consequently, there is a need for methods that can be used to solve such equations. The idea of using change of variable was introduced, which transforms an ordinary differential equation (ODE) into a simpler equation by Sophus Lie (1482). He used symmetry to solve differential equations by determining Lie groups of symmetries of a given ordinary differential equation. For an introduction to symmetry method for ODEs, see [Olver (1993), Bluman and Kumei (1989), Stephani (1989) and Hydon (2000)].

Meada (1987) had shown that difference equations of order one can be solved by Lie's method and he showed that the linearized symmetry condition (LSC) for such difference equation tends to a set of functional equations. Quisple and Sahdevan (1993) were interested by this method and they extended Meada's idea to higher order difference equations but in restricted form. Levi et al. (1997) expanded the linearized symmetry condition as a series in powers of u_n but the expression derived by them was complicated. Hydon (2000) introduced a method for obtaining the Lie symmetries and used it to reduce the order of the ordinary difference equation and to find the solution. Then, he applied this method to second order difference equations. Recently, symmetry methods have been extended to higher order difference equations [[4], [5], [7], [8]]. The idea consists in finding symmetries of the equations and use them to lower the order of the equation.

In this Thesis, we study the symmetry method for ordinary differential and difference equations. We investigate the exact solutions of sixth and eighth order nonlinear difference equations using a group of transformations (Lie symmetries).

This Thesis is organized as follows, in chapter one, we investigate symmetries methods for first order differential equations and we show how can we use symmetry to solve these equations. Then, we generalize the symmetry method for higher order difference equations and we show how can we use symmetry to solve non-linear higher order difference equations. In chapter two, we investigate symmetry methods for first and second order difference equations, and we show how can we use symmetry to solve these equations. We generalize the symmetry method for higher order difference equations and we show how can we use symmetry to solve non-linear higher order difference equations of third order. In chapter three we investigate the exact solutions of (4.1.1) and (4.2.1) non-linear difference equations using Lie symmetries.

In the future we want to study stability for difference equations (4.1.1) and (4.2.1).

2. PRELIMINARIES

2.1 Symmetry Methods for Differential Equations

2.1.1 Symmetry of Geometrical Object

To understand the concept of symmetries of ordinary differential equations, it is suitable to consider symmetries of objects. A symmetry of a geometrical object is an invertible transformation that maps the object to itself. The points of an object may be mapped to different points, but the object as a whole is unchanged by a symmetry. For example, consider the rotation of a regular octagon about its diameters ae, bf, cg, dh (see figure 2.1).

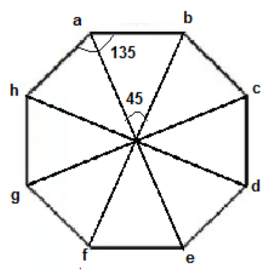

Fig. 2.1: Symmetries of an Octagon

We observe that the points themselves may change but the whole object stays as it is, so the transformation is a symmetry. Also, if the angle of rotation is an integer multiple of $\frac{\pi}{4}$, the object is mapped to itself, so the transformation is a symmetry (see figure 2.2).

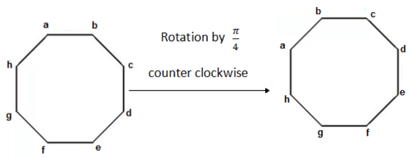

Fig. 2.2: Transformation by $\frac{\pi}{4}$

Moreover, every geometrical object has a trivial symmetry which is the transformation that maps every point to itself. In this example the rotation of an octagon by 2π is a trivial symmetry (see figure 2.3).

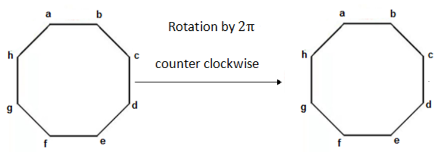

Fig. 2.3: Transformation by 2π

In addition, each symmetry of geometrical object has a unique inverse which is a symmetry. For example, let T denote a rotation of a regular octagon by $\frac{\pi}{2}$. Then the inverse of T (T^{-1}) is a rotation by $\frac{3\pi}{2}$ (see figure 2.4 and 2.5).

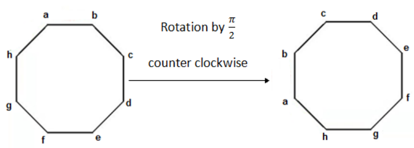

Fig. 2.4: Transformation by $\frac{\pi}{2}$

Definition 2.1. *[9] A symmetry is a diffeomorphism that maps the set of solutions of the ODE to itself.*

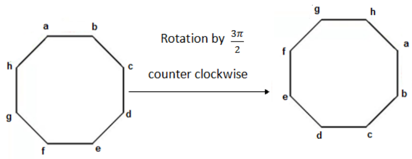

Fig. 2.5: Transformation by $\frac{3\pi}{2}$

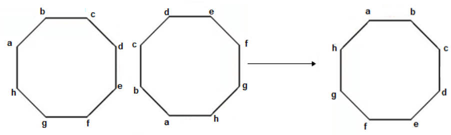

Fig. 2.6: Composition of transformation $\frac{\pi}{2}$ and $\frac{3\pi}{2}$

Symmetry must preserve the shape of the object, that is the distance between any two points of the object must be preserved. Therefore, the only transformations of Euclidean space consist of rotations, translations, and refections. We define symmetry as:

Definition 2.2. *[9] A transformation is a symmetry if it satisfies the following properties:*

1. *The transformation preserves the structure.*
2. *The transformation is a diffeomorphism, that is, it is a smooth invertible mapping whose inverse is also smooth.*
3. *The transformation maps the object to itself [e.g., a planar object in the (x, y) plane and its image in the $(\bar{x}, \bar{y})$ plane are indistinguishable].*

We restrict attention to transformations satisfying conditions 1 and 2. Such transformations are symmetries if they also satisfy condition 3, which is called the symmetry condition.

Example 2.1. [9] The ordinary differential equation (ODE)

$$\frac{dy}{dx} = 0 \tag{2.1.1}$$

has a symmetry

$$T_\alpha : (x, y) \to (\bar{x}, \bar{y}) = (x, y + \alpha), \alpha \in \mathbb{R}$$

The transformation is a translation in y by α.

- T_α preserves the structure, that is it preserves the distance between two points of the objects (solution curves).
- T_α is a smooth transformation of the plane and is invertible if its Jacobian is nonzero, so we consider the condition $\bar{x}_x \bar{y}_y - \bar{x}_y \bar{y}_x \neq 0$ (the inverse of translation by α is a translation by $-\alpha$)
- T_α maps a point (x, y) on one solution curve to point $(\bar{x}, \bar{y})$ on another solution, therefore

$$\frac{d\bar{y}}{d\bar{x}} = 0 \quad when \quad \frac{dy}{dx} = 0$$

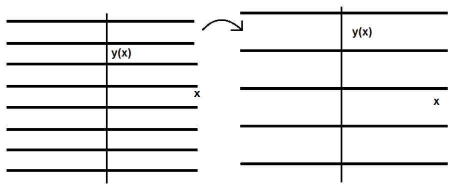

Fig. 2.7: The transformation is a translation in y by α

Definition 2.3. *[10] A group is a set H together with a group operation called group multiplication such that the following axioms are satisfied:*

- *Closure: $h_i \in H$ and $h_j \in H$, then $h_i * h_j \in H$.*
- *Associativity: For all $h_i \in H, h_j \in H, h_k \in H$, then $(h_i * h_j) * h_k = h_i * (h_j * h_k)$.*
- *Identity: There is a group operation I, called the identity operator with the property that $(h_i * I) = (I * h_i) = h_i$.*
- *Inverse: For each h_i in H there is an inverse, denoted by h_i^{-1} such that $(h_i * h_i^{-1}) = (h_i^{-1} * h_i) = I$.*

Theorem 2.1. *[7] Let H be the set of all symmetries of a geometrical object then H is a group under composition of the transformation.*

Proof. Let T_σ and T_α be two symmetries of an object then the composite transformations $T_\sigma T_\alpha$, and $T_\alpha T_\sigma$ are symmetries of this object, because they are invertible and they keep the object unchanged.
The trivial symmetry denoted by T_0 is the identity map, that is, for any $T_\alpha \in H$,

$$T_\alpha T_0 = T_0 T_\alpha = T_\alpha.$$

And for any $T_\alpha \in H$, the transformation that reverts the object to its original state, is the inverse of a transformation, that is,

$$T_\alpha T_\alpha^{-1} = T_\alpha^{-1} T_\alpha = T_0.$$

It's clear that, composition of transformations is associative, so H is group. ■

Definition 2.4. ***(Lie Group)***
Let (x, y) and $(\varphi(x, y, \alpha), \psi(x, y, \alpha))$ be two points in the Euclidean plane, and for α in $\mathbb{R}$, let $T_\alpha : (x, y) \mapsto (\varphi(x, y, \alpha), \psi(x, y, \alpha))$ be a transformation, depending on the parameter α, that takes point (x, y) to point $(\varphi(x, y, \alpha), \psi(x, y, \alpha))$. We say the set of transformations T_α is a (additive) Lie group H if the following conditions are satisfied:

1. *T_α is one to one, that is we assume that $\varphi(x, y, \alpha)$ and $\psi(x, y, \alpha)$ are functionally independent i.e., Jacobian does not vanish.*

$$\begin{vmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{vmatrix} \neq 0$$

   Also, T_α is onto that is, it is a transformation that carries any point (x, y) in the (x, y)-plane into a new position $(\bar{x}, \bar{y})$ such that $(\bar{x}, \bar{y}) = T_\alpha(x, y)$.

2. *Let T_{α_1} and T_{α_2} be transformations then*

$$T_{\alpha_1} \circ T_{\alpha_2} = T_{\alpha_1 + \alpha_2}$$

that is,

$$\begin{aligned} T_{\alpha_1}(\varphi(x,y,\alpha_2),\psi(x,y,\alpha_2)) &= (\varphi(x,y,\alpha_1+\alpha_2),\psi(x,y,\alpha_1+\alpha_2)) \\ &= T_{\alpha_1+\alpha_2} \end{aligned}$$

3. T_0 *is the identity transformation. That is,*

$$T_0 : (x,y) \mapsto (\varphi(x,y,0),\psi(x,y,0)) = (x,y)$$

It can be written as

$$T_0(x,y) = (x,y) = I$$

4. *For each* α_1 *there exists a unique* $\alpha_2 = -\alpha_1$ *such that* $T_{\alpha_2} \circ T_{\alpha_1} = T_0 = I$. *That is,*

$$\begin{aligned} T_{\alpha_2}(\varphi(x,y,\alpha_1),\psi(x,y,\alpha_1)) &= (\varphi(x,y,\alpha_1-\alpha_1),\psi(x,y,\alpha_1-\alpha_1)) \\ &= (\varphi(x,y,0),\psi(x,y,0)) \\ &= T_0 \\ &= I \end{aligned}$$

In addition to these four group properties, $(\varphi(x,y,\alpha),\psi(x,y),\alpha)$ is infinitely differentiable with respect to (x,y) and analytic with respect to α, we say H is a one-parameter Lie group or a Lie point transformation. For example, the infinite set of symmetries is a one-parameter Lie group.

Example 2.2. Consider the transformation

$$T_\alpha : (x,y) \mapsto (x+\alpha, y-\alpha)$$

1. T_α is one to one since

$$\begin{vmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

   Also T_α is onto since any point (x,y) in the (x,y)-plane is mapped into a new position $(\bar{x},\bar{y})$ such that

$$(\bar{x},\bar{y}) = T_\alpha(\bar{x}-\alpha,\bar{y}+\alpha)$$

2. Let T_{α_1} and T_{α_2} be transformations then

$$\begin{aligned} T_{\alpha_2} \circ T_{\alpha_1} &= T_{\alpha_2}(x+\alpha_1, y-\alpha_1) \\ &= (x+\alpha_1+\alpha_2, y-\alpha_1-\alpha_2) \\ &= (x+(\alpha_1+\alpha_2), y-(\alpha_1+\alpha_2)) \\ &= T_{\alpha_1+\alpha_2} \end{aligned}$$

3. T_0 is the identity transformation since

$$T_0 : (x, y) \mapsto (x+0, y-0) = (x, y)$$

So

$$T_0(x, y) = (x, y) = I$$

4. For each α_1 there exists a unique $\alpha_2 = -\alpha_1$ such that

$$\begin{aligned} T_{\alpha_2} \circ T_{\alpha_1} &= T_{\alpha_2}(x+\alpha_1, y-\alpha_2) \\ &= (x+\alpha_1+\alpha_2, y-\alpha_1-\alpha_2) \\ &= (x, y) \\ &= T_0 \\ &= I \end{aligned}$$

So the transformation T_α is a Lie group.

2.1.2 Symmetries of Differential Equations

A transformation of a differential equation is a symmetry if every solution of the transformed equation is a solution of the original equation and vice versa.

Definition 2.5. *[9] Consider the following transformation*

$$T_\alpha : x^s \mapsto \bar{x}^s(x^1, \cdots, x^N; \alpha), s = 1, \cdots, N.$$

where α is a real parameter. Then T_α is one-parameter local Lie group if the following conditions are satisfied:

1. *T_0 is the trivial symmetry, so that $\bar{x}^s = x^s$ when $\alpha = 0$.*

2. *T_α is a symmetry for every α in some neighbourhood of zero.*

3. *$T_\alpha T_\beta = T_{\alpha+\beta}$ for every α, β sufficiently close to zero.*

4. *Each $\bar{x}^s$ may be represented by a Taylor series in α (in some neighbourhood of $\alpha = 0$), so*

$$\bar{x}^s(x^1, \cdots, x^N; \alpha) = x^s + \alpha\xi^s(x^1, \cdots, x^N) + O(\alpha^2), s = 1, \cdots, N.$$

The term"local" is used because the conditions need only apply in some neighbourhood of $\alpha = 0$. Also, the maximum size of the neighbourhood may depend on x^s, $s = 1, \cdots, N$. A local Lie group may not be a group, it needs only satisfy the group axioms for sufficiently small parameter values.
From condition 2 we have $T_\alpha^{-1} = T_{-\alpha}$ when $|\alpha|$ is sufficiently small.
A one parameter Lie group of symmetries of a differential equation will depend continuously on the parameter. For simplicity, we call symmetries that belong to a one parameter local Lie group, Lie symmetries.

Lie groups may not necessary be defined over the entire real number plane. We will be dealing with local groups, that is, the group action may not be defined over the whole plane. The following example illustrates this:

Example 2.3. [9] The Riccati equation

$$\frac{dy}{dx} = \frac{y+1}{x} + \frac{y^2}{x^3} \tag{2.1.2}$$

has a symmetry

$$T_\alpha : (x, y) \mapsto (\bar{x}, \bar{y}) = (\frac{x}{1-\alpha x}, \frac{y}{1-\alpha x}) \tag{2.1.3}$$

which is defined only if $\alpha < \frac{1}{x}$ when $x > 0$ and $\alpha > \frac{1}{x}$ when $x < 0$. If $\alpha = \frac{1}{x}$ then the transfomation is undefined. If $\alpha = 0$ then

$$(\bar{x}, \bar{y}) = (x, y)$$

which is the identity transformation. Therefore, the interval on which T_α is defined must include the origin. If $x > 0$ and $\alpha > \frac{1}{x}$ then the identity is not included in this interval. Similarly, if $x < 0$ and $\alpha < \frac{1}{x}$ then the origin is not included. Therefore, in order for the group to have an identity, it can only be defined when $\alpha < \frac{1}{x}$ for $x > 0$ and $\alpha > \frac{1}{x}$ for $x < 0$. This means that, for a fixed α, the domain of T_α is $-\frac{1}{\alpha} < x < \frac{1}{\alpha}$ and $y \in \mathbb{R}$. The range is $-\frac{1}{2\alpha} < \bar{x}$ and $\bar{y} \in \mathbb{R}$.

The symmetry T_α defined by (2.1.3) satisfies the conditions:

1. T_α is one to one, if (x_1, y_1) and (x_2, y_2) are both mapped to the same point $(\bar{x}, \bar{y})$ then

$$\bar{x} = \frac{x_1}{1 - \alpha x_1} = \frac{x_2}{1 - \alpha x_2} \tag{2.1.4}$$

Then,

$$x_1(1 - \alpha x_2) = x_2(1 - \alpha x_1)$$

$$x_1 - \alpha x_1 x_2 = x_2 - \alpha x_2 x_1$$

we have

$$x_1 = x_2$$

Similarly,

$$\bar{y} = \frac{y_1}{1 - \alpha x_1} = \frac{y_2}{1 - \alpha x_2}$$

we have

$$y_1 = y_2$$

Also T_α is onto since if

$$\bar{x} = \frac{x}{1 - \alpha x}$$

Then

$$x = \bar{x} - \alpha\bar{x}x$$

we have

$$x = \frac{\bar{x}}{1+\alpha\bar{x}}$$

and consider

$$\bar{y} = \frac{y}{1-\alpha x}$$

Then

$$y = \bar{y}\left(1 - \frac{\alpha\bar{x}}{1+\alpha\bar{x}}\right) = \frac{\bar{y}}{1+\alpha\bar{x}}$$

Therefore, there is a point (x, y) that corresponds to every point $(\bar{x}, \bar{y})$.

2. Let T_{α_1} and T_{α_2} be two transformations then

$$\begin{aligned} T_{\alpha_2} \circ T_{\alpha_1} &= T_{\alpha_2}\left(\frac{x}{1-\alpha_1 x}, \frac{y}{1-\alpha_1 x}\right) \\ &= \left(\frac{\frac{x}{1-\alpha_1 x}}{1-\alpha_2\frac{x}{1-\alpha_1 x}}, \frac{\frac{y}{1-\alpha_1 x}}{1-\alpha_2\frac{x}{1-\alpha_1 x}}\right) \\ &= \left(\frac{x}{1-(\alpha_1+\alpha_2)x}, \frac{y}{1-(\alpha_1+\alpha_2)x}\right) \\ &= T_{\alpha_1+\alpha_2} \end{aligned}$$

The domain of this composition is $-\frac{1}{\alpha_1+\alpha_2} < x < \frac{1}{\alpha_1+\alpha_2}$ and $y \in \mathbb{R}$ and the range is $-\frac{1}{2(\alpha_1+\alpha_2)} < \bar{x}$ and $\bar{y} \in \mathbb{R}$.

3. T_0 is the identity transformation since

$$T_0 : (x, y) \mapsto (\bar{x}, \bar{y}) = (\frac{x}{1-0}, \frac{y}{1-0}) = (x, y)$$

4. For all $\alpha_1 \in \mathbb{R}$ there exists a unique $\alpha_2 = -\alpha_1$ such that

$$
\begin{aligned}
T_{\alpha_2} \circ T_{\alpha_1} &= T_{\alpha_2}\left(\frac{x}{1-\alpha_1 x}, \frac{y}{1-\alpha_1 x}\right) \\
&= \left(\frac{\frac{x}{1-\alpha_1 x}}{1-\alpha_2 \frac{x}{1-\alpha_1 x}}, \frac{\frac{y}{1-\alpha_1 x}}{1-\alpha_2 \frac{x}{1-\alpha_1 x}}\right) \\
&= \left(\frac{x}{1-(\alpha_1-\alpha_1)x}, \frac{y}{1-(\alpha_1-\alpha_1)x}\right) \\
&= (x, y) \\
&= T_0 \\
&= I
\end{aligned}
$$

So (2.1.3) is a Lie group defined only when $\alpha < \frac{1}{x}$ if $x > 0$ and $\alpha > \frac{1}{x}$ if $x < 0$.

2.1.3 The Symmetry Condition

Consider the first-order differential equation of the form

$$\frac{dy}{dx} = \omega(x, y) \tag{2.1.5}$$

The symmetry condition is important to find any symmetry that maps the set of solution curves in the (x, y) plan to an identical set of curves in the $(\bar{x}, \bar{y})$ plane. So the symmetry equation for (2.1.5) is

$$\frac{d\bar{y}}{d\bar{x}} = \omega(\bar{x}, \bar{y}) \quad when \quad \frac{dy}{dx} = \omega(x, y) \tag{2.1.6}$$

We can write this condition using the total derivative operator

$$\frac{d\bar{y}}{d\bar{x}} = \frac{D_x\bar{y}}{D_x\bar{x}} = \frac{\bar{y}_x + \frac{dy}{dx}\bar{y}_y}{\bar{x}_x + \frac{dy}{dx}\bar{x}_y} = \omega(\bar{x}, \bar{y}) \quad when \quad \frac{dy}{dx} = \omega(x, y) \tag{2.1.7}$$

where the total derivative operator is

$$D_x = \frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y} + \frac{d^2y}{dx^2}\frac{\partial}{\partial y'} + \cdots$$

From equation (2.1.5), we get

$$\frac{D_x\bar{y}}{D_x\bar{x}} = \frac{\bar{y}_x + \omega(x,y)\bar{y}_y}{\bar{x}_x + \omega(x,y)\bar{x}_y} = \omega(\bar{x},\bar{y}) \tag{2.1.8}$$

Example 2.4. [9] The differential equation

$$\frac{dy}{dx} = y^2e^{-x} + y + e^x \tag{2.1.9}$$

has a symmetry

$$(\bar{x},\bar{y}) = (x + 2\alpha, ye^{2\alpha}) \tag{2.1.10}$$

we substitute equation (2.1.9) into the symmetry condition, we get

$$\frac{\bar{y}_x + \left(y^2e^{-x} + y + e^x\right)\bar{y}_y}{\bar{x}_x + \left(y^2e^{-x} + y + e^x\right)\bar{x}_y} = \omega(\bar{x},\bar{y}) \tag{2.1.11}$$

where

$$\begin{aligned} \bar{y}_x &= 0 \\ \bar{y}_y &= e^{2\alpha} \\ \bar{x}_x &= 1 \\ \bar{x}_y &= 0 \end{aligned}$$

So

$$(y^2e^{-x} + y + e^x)e^{2\alpha} = \omega(\bar{x},\bar{y}) \tag{2.1.12}$$

On the left hand side of equation (2.1.12), we have

$$\begin{aligned} (y^2e^{-x} + y + e^x)e^{2\alpha} &= y^2e^{-x}e^{2\alpha} + ye^{2\alpha} + e^xe^{2\alpha} \\ &= y^2e^{-x}e^{2\alpha}\frac{e^{2\alpha}}{e^{2\alpha}} + ye^{2\alpha} + e^xe^{2\alpha} \\ &= \bar{y}^2e^{-\bar{x}} + \bar{y} + e^{\bar{x}} \\ &= \omega(\bar{x},\bar{y}) \end{aligned}$$

On the right hand side of equation (2.1.12), we have

$$\begin{aligned}\omega(\bar{x},\bar{y}) &= \bar{y}^2 e^{-\bar{x}} + \bar{y} + e^{\bar{x}}\\ &= y^2 e^{4\alpha} e^{-x-2\alpha} + y e^{2\alpha} + e^{x+2\alpha}\\ &= y^2 e^{-x} e^{2\alpha} + e^{2\alpha} + e^{x} e^{2\alpha}\end{aligned}$$

So the symmetry condition is satisfied and (2.1.10) is a symmetry of equation (2.1.9).

Example 2.5. Consider the Riccati equation

$$\frac{dy}{dx} = xy^2 - \frac{2y}{x} - \frac{1}{x^3},\ x \neq 0 \tag{2.1.13}$$

has a symmetry

$$(\bar{x},\bar{y}) = (e^{\alpha}x, e^{-2\alpha}y) \tag{2.1.14}$$

we substitute equation (2.1.13) into the symmetry condition, we get

$$\frac{\bar{y}_x + \left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)\bar{y}_y}{\bar{x}_x + \left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)\bar{x}_y} = \omega(\bar{x},\bar{y}) \tag{2.1.15}$$

where

$$\begin{aligned}\bar{y}_x &= 0\\ \bar{y}_y &= e^{-2\alpha}\\ \bar{x}_x &= e^{\alpha}\\ \bar{x}_y &= 0\end{aligned}$$

So

$$\frac{\left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)e^{-2\alpha}}{e^{\alpha}} = \omega(\bar{x},\bar{y}) \tag{2.1.16}$$

On the left hand side of equation (2.1.16), we have

$$\begin{aligned}\frac{\left(xy^2 - \frac{2y}{x} - \frac{1}{x^3}\right)e^{-2\alpha}}{e^{\alpha}} &= \frac{xy^2e^{-2\alpha}}{e^{\alpha}} - \frac{\frac{2y}{x}e^{-2\alpha}}{e^{\alpha}} - \frac{\frac{1}{x^3}e^{-2\alpha}}{e^{\alpha}} \\ &= \frac{xy^2e^{-2\alpha}}{e^{\alpha}}\frac{e^{\alpha}}{e^{\alpha}} - \frac{2\bar{y}}{\bar{x}} - \frac{1}{e^{3\alpha}x^3} \\ &= \bar{x}\bar{y} - \frac{2\bar{y}}{\bar{x}} - \frac{1}{\bar{x}^3} \\ &= \omega(\bar{x}, \bar{y})\end{aligned}$$

On the right hand side of equation (2.1.16), we have

$$\begin{aligned}\omega(\bar{x}, \bar{y}) &= \bar{x}\bar{y}^2 - \frac{2\bar{y}}{\bar{x}} - \frac{1}{\bar{x}^3} \\ &= e^{\alpha}xe^{-4\alpha}y - \frac{2e^{-2\alpha}y}{e^{\alpha}x} - \frac{1}{e^{3\alpha}x^3} \\ &= xye^{-3\alpha} - \frac{2y}{x}e^{-3\alpha} - \frac{1}{x^3}e^{-3\alpha} \\ &= (xy - \frac{2y}{x} - \frac{1}{x^3})e^{-3\alpha}\end{aligned}$$

So the symmetry condition is satisfied and (2.1.14) is a symmetry of equation (2.1.13).

2.1.4 A Change of Coordinates

Any ODE that has a symmetry of the form

$$(\bar{x}, \bar{y}) = (x, y + \alpha)$$

can be reduced to quadrature. This means that the differential equation can be solved by an integrating technique. For all α in some neighbourhood of zero, the symmetry condition reduces to

$$\omega(x, y) = \omega(x, y + \alpha) \tag{2.1.17}$$

Differentiate equation (2.1.17) with respect to α at $\alpha = 0$

$$\frac{\partial}{\partial \alpha}\omega(x,y) = \frac{\partial}{\partial \alpha}\omega(x, y+\alpha)$$
$$0 = \frac{\partial \omega}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial \omega}{\partial y}\frac{\partial y}{\partial \alpha}$$

From equation (2.1.17), we get

$$\frac{\partial x}{\partial \alpha} = 0$$

and

$$\frac{\partial y}{\partial \alpha} = 1$$

So

$$0 = \frac{\partial \omega}{\partial y}$$

Thus the original ODE is a function of x only and then

$$\frac{dy}{dx} = \omega(x)$$

and

$$y = \int \omega(x)dx + c$$

While a symmetry of the form in Equation (2.1.17) does not exist in cartesian coordinates for all differential equations, it is possible to find such a symmetry in a different coordinate system.

Example 2.6. Consider the ODE

$$\frac{dy}{dx} = \frac{y^3 + x^2 y - y - x}{xy^2 + x^3 + y - x} \tag{2.1.18}$$

Equation (2.1.18) is difficult to solve in cartesian coordinates, to solve it we rewrite the equation in terms of polar coordinates. Let

$$x = r\cos\theta, \;\; y = r\sin\theta \tag{2.1.19}$$

then

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta = \cos\theta dr - r\sin\theta d\theta \tag{2.1.20}$$

and

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta = \sin\theta dr + r\cos\theta d\theta \tag{2.1.21}$$

Substituting (2.1.19) into (2.1.18), we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{r^3\sin^3\theta + r^3\cos^2\theta\sin\theta - r\sin\theta - r\cos\theta}{r^3\cos\theta\sin^2\theta + r^3\cos^3\theta + r\sin\theta - r\cos\theta} \\
&= \frac{r^3\sin\theta\Big(sin^2\theta + \cos^2\theta\Big) - r\sin\theta - r\cos\theta}{r^3\cos\theta\Big(\sin^2\theta + \cos^2\theta\Big) + r\sin\theta - r\cos\theta} \\
&= \frac{r^3\sin\theta - r\sin\theta - r\cos\theta}{r^3\cos\theta + r\sin\theta - r\cos\theta} \\
&= \frac{r^2\sin\theta - \sin\theta - \cos\theta}{r^2\cos\theta + \sin\theta - \cos\theta} \\
&= \frac{\sin\theta(r^2-1) - \cos\theta}{\cos\theta(r^2-1) + \sin\theta}
\end{aligned} \tag{2.1.22}$$

Substituting (2.1.20) and (2.1.21) into (2.1.22), we get

$$\frac{dy}{dx} = \frac{\sin\theta dr + r\cos\theta d\theta}{\cos\theta dr - r\sin\theta d\theta} = \frac{\sin\theta(r^2-1) - \cos\theta}{\cos\theta(r^2-1) + \sin\theta}$$

Cross multiplication this equation, we get

$$\Big(\sin\theta dr + r\cos\theta d\theta\Big)\Big(\cos\theta(r^2-1) + \sin\theta\Big) = \Big(\cos\theta dr - r\sin\theta d\theta\Big)\Big(\sin\theta(r^2-1) - \cos\theta\Big)$$

Solving this equation to get $\frac{dr}{d\theta}$

$$\begin{aligned}
&\sin\theta\cos\theta(r^2-1)dr + \sin^2\theta dr + r\cos^2\theta(r^2-1)d\theta + r\cos\theta\sin\theta d\theta \\
&= \cos\theta\sin\theta(r^2-1)dr - \cos^2\theta dr - r\sin^2\theta(r^2-1)d\theta + r\sin\theta\cos\theta d\theta
\end{aligned}$$

So

$$\begin{aligned}
\sin^2\theta dr + r\cos^2\theta(r^2-1)d\theta &= -\cos^2\theta dr - r\sin^2\theta(r^2-1)d\theta \\
\sin^2\theta dr + \cos^2\theta dr &= -r\cos^2\theta(r^2-1)d\theta - r\sin^2\theta(r^2-1)d\theta \\
dr &= -r\Big(\cos^2\theta(r^2-1)d\theta + \sin^2\theta(r^2-1)\Big)d\theta
\end{aligned}$$

we have

$$\frac{dr}{d\theta} = -r(r^2-1)\Big(\cos^2\theta + \sin^2\theta\Big)$$
$$= r(1-r^2) \tag{2.1.23}$$

This equation is separable in this coordinate system

$$\int \frac{dr}{r(1-r^2)} = \int d\theta$$

we have

$$\int \frac{1}{r}dr - \frac{1}{2}\int \frac{1}{1+r}dr + \frac{1}{2}\int \frac{1}{1-r}dr = \theta$$

So

$$\ln r - \frac{1}{2}\ln(1+r) - \frac{1}{2}\ln(1-r) = \theta$$
$$\ln \frac{r}{\sqrt{(1-r^2)}} = \theta$$

Equation (2.1.23) has the symmetry

$$(\bar{r}, \bar{\theta}) = (r, \theta+\alpha) \tag{2.1.24}$$

The symmetry condition for (2.1.24) is

$$\frac{d\bar{r}}{d\bar{\theta}} = \frac{\bar{r}_\theta + \frac{dr}{d\theta}\bar{r}_r}{\bar{\theta}_\theta + \frac{dr}{d\theta}\bar{\theta}_r}$$

where

$$\bar{r}_r = 1$$
$$\bar{r}_\theta = 0$$
$$\bar{\theta}_\theta = 1$$
$$\bar{\theta}_r = 0$$

we have

$$\frac{d\bar{r}}{d\bar{\theta}} = r(1-r^2)$$

So the symmetry condition is satisfied. Figure (2.8) shows some of the solution curves for Equation (2.1.23). The solution curves are rotational symmetries. When written in polar coordinates, the symmetry for Equation (2.1.23) indicates that the rotational symmetries are translations in θ.

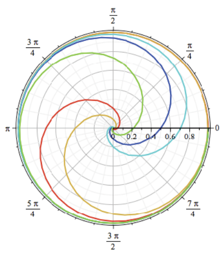

Fig. 2.8: Solution curves of equation (2.1.23).

2.1.5 Orbits

The following definition is important for solving differential equation by using symmetry methods.

Definition 2.6. *[9] Consider a particular point (x, y) and the action of additive Lie group*

$$T_\alpha : (x, y) \mapsto (\bar{x}, \bar{y}) = (\bar{x}(x, y; \alpha), \bar{y}(x, y; \alpha))$$

As $\alpha \in \mathbb{R}$, the point $(\bar{x}, \bar{y})$ moves in the plane along a continuous curve. This curve is called the orbit of (x, y) under the group. If the Lie group is a nontrivial symmetry group of a differential equation, then an orbit of the group takes a continuous path transverse to solution curves of the differential equation $\frac{dy}{dx} = \omega(x, y)$. An orbit of solution curves is a continuous family. Along this orbit, changes in α map solution curves to other solution curves.

Example 2.7. In example (2.1) the orbit of a point on a solution curve of this differential equation are vertical lines under the symmetry. For instance, the orbit of the point (1, 0) includes $\{(1, 1), (1, 2), \cdots\}$ (see figure 2.9).

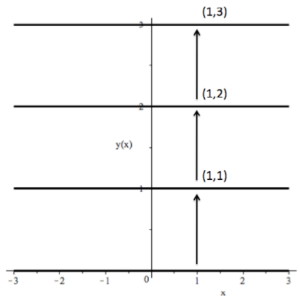

Fig. 2.9: Solutions to Equation (2.1) and orbit of the point $(1, 0)$.

2.1.6 Tangent Vectors

$\frac{D_x\bar{y}}{D_x\bar{x}} = \omega(\bar{x}, \bar{y})$ is nonlinear PDE that can be solved by linearizing it to find the coordinate system. The tangent vectors to an orbit under a given symmetry are important to determine the new coordinate system. The tangent vectors to the orbit at any point $(\bar{x}, \bar{y})$ are described by the tangent vector in the x-direction, denoted by $\xi(\bar{x}, \bar{y})$ and the tangent vector in the y-direction, denoted by $\eta(\bar{x}, \bar{y})$. Therefor,

$$\frac{d\bar{x}}{d\alpha} = \xi(\bar{x}, \bar{y})$$

and

$$\frac{d\bar{y}}{d\alpha} = \eta(\bar{x}, \bar{y})$$

At the initial point (x, y), when α equals 0, we have

$$\left(\left.\frac{d\bar{x}}{d\alpha}\right|_{\alpha=0}, \left.\frac{d\bar{y}}{d\alpha}\right|_{\alpha=0} \right) = (\xi(x, y), \eta(x, y)) \tag{2.1.25}$$

The tangent vectors are useful for finding invariant solution curves. An invariant solution curve is always mapped to itself under a symmetry. The points on an invariant solution curve are mapped either to themselves or other points on the same curve. Therefore, the orbit of a noninvariant point on an invariant solution curve is the solution curve itself. When a solution curve is invariant, this means that the derivative at the point (x, y) will point in the same direction as the tangent vectors to the orbit. As α varies, the point is mapped to another point on the same solution curve, rather than a different solution curve. Therefore,

$$\frac{dy}{dx} = \frac{\eta(x,y)}{\xi(x,y)}$$

To find the tangent vector field of the group of orbits, we expand $\bar{x}$, $\bar{y}$ and $\omega(\bar{x}, \bar{y})$ in Taylor series expansion around $\alpha = 0$

$$\bar{x} = x + \alpha\tfrac{d\bar{x}}{d\alpha}|_{\alpha=0} + O(\alpha^2) = x + \alpha\xi(x,y) + O(\alpha^2) \tag{2.1.26}$$

$$\bar{y} = x + \alpha\tfrac{d\bar{x}}{d\alpha}|_{\alpha=0} + O(\alpha^2) = y + \alpha\eta(x,y) + O(\alpha^2) \tag{2.1.27}$$

$$\begin{aligned} \omega(\bar{x},\bar{y}) &= \omega(x,y) + \alpha\frac{d\omega(\bar{x},\bar{y})}{d\alpha}|_{\alpha=0} + O(\alpha^2) \\ &= \omega(x,y) + \alpha\Big(\omega_x(x,y)\xi(x,y) + \omega_y(x,y)\eta(x,y)\Big) + O(\alpha^2) \end{aligned} \tag{2.1.28}$$

where $O(\alpha^2)$ describes the error function for the Taylor series expansions of $\bar{x}$, $\bar{y}$, and $\omega(\bar{x}, \bar{y})$ we ignore terms of α^2 or higher. For simplicity, $\xi(x, y)$ will be denoted merely as ξ and $\eta(x, y)$ as η.

From equation (2.1.8) we calculate $D_x\bar{x}$ and $D_x\bar{y}$ by using Taylor series expansion of $\bar{x}$ and $\bar{y}$ respectively, we have

$$\begin{aligned} D_x\bar{x} &= D_x(x + \alpha\xi + O(\alpha^2)) \\ &= 1 + \alpha\xi_x + y'\alpha\xi_y + O(\alpha^2) \\ &= 1 + \alpha(\xi_x + \omega\xi_y) + O(\alpha^2) \end{aligned} \tag{2.1.29}$$

and

$$\begin{aligned} D_x\bar{y} &= D_x(y + \alpha\eta + O(\alpha^2)) \\ &= \alpha\eta_x + y'(1 + \alpha\eta_y) + O(\alpha^2) \\ &= \omega + \alpha(\eta_x + \omega\eta_y) + O(\varepsilon^2) \end{aligned} \tag{2.1.30}$$

Substituting (2.1.29) and (2.1.30) into (2.1.8), we get

$$\frac{\omega + \alpha(\eta_x + \omega\eta_y)}{1 + \alpha(\xi_x + \omega\xi_y)} = \omega + \alpha(\omega_x\xi + \omega_y\eta)$$

Multiplying by the denominator

$$\omega + \alpha(\eta_x + \omega\eta_y) = (1 + \alpha(\xi_x + \omega\xi_y))(\omega + \alpha(\omega_x\xi + \omega_y\eta))$$

Disregarding terms of α^2 or higher, we obtain

$$\eta_x + (\eta_y - \xi_x)\omega - \xi_y\omega^2 - (\xi\omega_x + \eta\omega_y) = 0 \tag{2.1.31}$$

This equation is called the linearized symmetry condition for first order differential equations ODE. This condition is necessary to solve the ODE but sometimes it is difficult to solve so we can use an appropriate "anatzs", that is, to place some additional constraints upon ξ and η.

The linearized symmetry condition can be rewritten in terms of the reduced characteristic, Q is defined by Hydon [9] as

$$Q(x, y, y') = \eta - y'\xi$$

but $\frac{dy}{dx} = \omega(x, y)$, we have

$$Q = \eta - \omega(x, y)\xi$$

as follows

$$Q_x + \omega(x, y)Q_y - \omega_y(x, y)Q = 0 \tag{2.1.32}$$

If Q satisfies (2.1.32) then each solution of (2.1.32) corresponds to infinitely many Lie groups. It follows that

$$(\xi, \eta) = (\xi, Q + \omega(x, y)\xi)$$

is a tangent vector field of a one-parameter group, for any function ξ. If $Q = 0$ then every solution curve is invariant under that symmetry (trivial symmetry). However, if not then the nontrivial symmetries can be found from (2.1.32) by using the method of characteristics. The characteristic equations are

$$\frac{dx}{1} = \frac{dy}{\omega(x, y)} = \frac{dQ}{\omega_y(x, y)Q} \tag{2.1.33}$$

Example 2.8. In example (2.5) the tangent vectors are

$$\xi(x, y) = x$$

and

$$\eta(x, y) = -2y$$

The reduced characteristic is

$$Q(x, y) = -2y - (xy^2 - \frac{2y}{x} - \frac{1}{x^3})x \tag{2.1.34}$$

$$= -x^2y^2 + \frac{1}{x^2} \tag{2.1.35}$$

$Q(x, y) = 0$ when $y = \pm\frac{1}{x^2}$. Therefore, the symmetry (2.1.14) acts nontrivially on all the solution curves of (2.1.13) except for $y = \frac{1}{x^2}$ and $y = -\frac{1}{x^2}$.

2.1.7 Canonical Coordinats

The aim of changing to a different coordinate system is to make a differential equation easier to solve. If the ODE (2.1.5) has a symmetry of the form $(x, y) = (x, y+\alpha)$, it can be solved by an integrating technique. However, not all differential equations have a symmetry of this form in cartesian coordinates. Therefore, one can change to a new coordinate system in $(r(x,y), s(x,y))$ to get a symmetry

$$T_\alpha : (r,s) \mapsto (\bar{r},\bar{s}) = (r, s+\alpha)$$

The tangent vectors at (r,s) when $\alpha = 0$ are

$$\left.\frac{d\bar{r}}{d\alpha}\right|_{\alpha=0} = 0$$

and

$$\left.\frac{d\bar{s}}{d\alpha}\right|_{\alpha=0} = 1$$

Applying total derivative with respect to α at $\alpha = 0$, we obtain

$$\begin{aligned}\left.\frac{d\bar{r}}{d\alpha}\right|_{\alpha=0} &= \left.\frac{d\bar{r}}{dx}\frac{dx}{d\alpha}\right|_{\alpha=0} + \left.\frac{d\bar{r}}{dy}\frac{dy}{d\alpha}\right|_{\alpha=0} \\ &= \frac{dr}{dx}\xi(x,y) + \frac{dr}{dy}\eta(x,y) = 0 \end{aligned} \tag{2.1.36}$$

and

$$\begin{aligned}\left.\frac{d\bar{s}}{d\alpha}\right|_{\alpha=0} &= \left.\frac{d\bar{s}}{dx}\frac{dx}{d\alpha}\right|_{\alpha=0} + \left.\frac{d\bar{s}}{dy}\frac{dy}{d\alpha}\right|_{\alpha=0} \\ &= \frac{ds}{dx}\xi(x,y) + \frac{ds}{dy}\eta(x,y) = 1 \end{aligned} \tag{2.1.37}$$

This equation can be written as

$$r_x\xi(x,y) + r_y\eta(x,y) = 0 \tag{2.1.38}$$

and

$$s_x\xi(x,y) + s_y\eta(x,y) = 1 \tag{2.1.39}$$

Equations (2.1.38) and (2.1.39) are first order linear partial differential equations for $r(x,y)$ and $s(x,y)$, respectively. They can be solved by the method of characteristics. Consider equation (2.1.38) in $r(x,y)$, the solution of (2.1.38) can be represented as surfaces $r(x,y)$ that satisfy

$$\langle r_x, r_y, -1\rangle \cdot \langle \xi(x,y), \eta(x,y), 0\rangle = 0$$

We know that the normal to the surface $r(x,y)$ is given by $\langle r_x, r_y, -1\rangle$. Therefore, if the vector $\langle \xi(x,y), \eta(x,y), 0\rangle$ is perpendicular to $\langle r_x, r_y, -1\rangle$ at each point then the vector $\langle \xi(x,y), \eta(x,y), 0\rangle$ lies in the tangent plane to the surface $r(x,y)$.

Consider a curve C parameterized by t such that at each point on the curve C, the vector $\langle \xi(x(t),y(t)), \eta(x(t),y(t)), 0\rangle$ is tangent to the curve. The curve C satisfies the following system of ODEs:

$$\frac{dx}{dt} = \xi(x(t),y(t)), \frac{dy}{dt} = \eta(x(t),y(t)), \frac{dr}{dt} = 0$$

So the characteristic equations for (2.1.38) are

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)} \tag{2.1.40}$$

To find the characteristic equations for equation (2.1.39), we have

$$\frac{dx}{dt} = \xi(x(t),y(t)), \frac{dy}{dt} = \eta(x(t),y(t)), \frac{ds}{dt} = 1 \tag{2.1.41}$$

So

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)} = \frac{ds}{1} \tag{2.1.42}$$

Now consider the function $\phi(x,y)$, the first integral of a differential equation:

$$\frac{dy}{dx} = f(x,y) \tag{2.1.43}$$

The first integral of an ODE (2.1.43) is a nonconstant function $\phi(x,y)$ whose value is constant on any solution $y = y(x)$ of the ODE (2.1.43). Applying the total derivative operator to $\phi(x,y)$, we get

$$\phi_x + f(x,y)\phi_y = 0, \ \ \phi_y \neq 0 \tag{2.1.44}$$

The general solution of (2.1.43) is

$$\phi(x,y) = c$$

If we divide Equation (2.1.36) by $\xi(x,y)$, we obtain

$$\frac{dr}{dx}\frac{\xi(x,y)}{\xi(x,y)} + \frac{dr}{dy}\frac{\eta(x,y)}{\xi(x,y)} = 0$$

so

$$r_x + \frac{dr}{dy} r_y = 0$$

Comparing this equation with equation (2.1.44), we find that $r(x,y)$ is a first integral of

$$\frac{dy}{dx} = \frac{\eta(x,y)}{\xi(x,y)}, \ \xi(x,y) \neq 0 \tag{2.1.45}$$

Therefore, in order to find r, one can solve Equation (2.1.45). Because $r(x,y)$ is a first integral of Equation (2.1.45)

$$r(x,y) = c = \phi(x,y), \ \textit{where } c \textit{ is constant}$$

To find s, one can use equation (2.1.42), we have

$$s = \int \frac{dy}{\eta(x,y)} = \int \frac{dx}{\xi(x,y)}$$

There is a special case when $\xi(x,y) = 0$. If $\xi(x,y) = 0$ and $\eta(x,y) \neq 0$ then

$$(r,s) = \left(x, \int \frac{dy}{\eta(x,y)}\right)\Bigg|_{r=x}$$

Example 2.9. In example (2.3) the one parameter Lie group is

$$(\bar{x},\bar{y}) = \left(\frac{x}{1-\alpha x}, \frac{y}{1-\alpha x}\right) \tag{2.1.46}$$

From equation (2.1.25) the tangent vectors is

$$\xi(x,y) = \frac{x^2}{1-\alpha x}\Bigg|_{\alpha=0} = x^2$$

and

$$\eta(x,y) = \left.\frac{xy}{1-\alpha x}\right|_{\alpha=0} = xy$$

So

$$(\xi(x,y), \eta(x,y)) = (x^2, xy)$$

From equation (2.1.45) we substitute $\xi(x,y)$ and $\eta(x,y)$ to find $r(x,y)$

$$\frac{dy}{dx} = \frac{y}{x}$$

This equation is separable

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

we get

$$\ln y = \ln x + c_1$$

which simplifies to

$$y = cx, \ \ where\ c = e^{c_1}$$

when we solve for c, we get

$$c = r(x,y) = \frac{y}{x}$$

using equation (2.1.42) to find s, we get

$$s(x,y) = \int \frac{dx}{x^2} = \frac{-1}{x}$$

So the canonical coordinates are

$$(r(x,y), s(x,y)) = \left(\frac{y}{x}, \frac{-1}{x}\right)$$

2.1.8 A New Way to Solve Differential Equations

To solve a given ODE, we write the differential equation of the canonical coordinates in terms r and s. Then, we can set the solution back into cartesian coordinates. To find $\frac{ds}{dr}$, apply the total derivative operator to get

$$\frac{ds}{dr} = \frac{s_x + \omega(x,y)s_y}{r_x + \omega(x,y)r_y} \tag{2.1.47}$$

This equation is written in terms of x and y. To write it in terms of r and s solve the canonical $r(x,y)$ and $s(x,y)$ for x and y then simplify and translate the solution into the cartesian coordinate.

Example 2.10. In example (2.4), the linearized symmetry condition is

$$\eta_x + (\eta_y - \xi_x)\omega - \xi_y\omega^2 - \xi(-e^{-x}y^2 + e^x) - \eta(2ye^{-x} + 1) = 0$$

It is necessary to solve this equation for ξ and η. We can make an ansatz((trial solution)) about ξ and η. Suppose $\xi = 1$ and η is a function of y only. Therefore,

$$\eta_y(y^2e^{-x} + y + e^x) - \xi(-e^{-x}y^2 + e^x) - \eta(2ye^{-x} + 1) = 0$$

Simplifying this equation, we get

$$e^{-x}y(\eta_y y + y - 2\eta) + e^x(\eta_y - 1) + \eta_y y - \eta = 0$$

we have

$$\eta_y y + y - 2\eta = 0 \tag{2.1.48}$$

$$\eta_y - 1 = 0 \tag{2.1.49}$$

$$\eta_y y - \eta = 0 \tag{2.1.50}$$

Equation (2.1.49) can be solved by separation of variable, we obtain

$$\eta = y$$

Now we can find the canonical coordinates r and s. To find r solve equation (2.1.45), we get

$$\frac{dy}{dx} = \frac{y}{1} = y$$

we have

$$\ln y = x + c_1, \ where \ c_1 \in \mathbb{R}$$

Therefore,

$$y = ce^x, \ where \ c = e^{c_1} \in \mathbb{R}$$

So

$$r = \frac{y}{e^x}$$

To find s, solve equation (2.1.42), we get

$$s = \int dx = x$$

Therefore, the canonical coordinates are

$$(r, s) = \left(\frac{y}{e^x}, x\right)$$

Substituting the canonical coordinates into equation (2.1.47)

$$\begin{aligned}\frac{ds}{dr} &= \frac{1}{-ye^{-x} + e^{-x}(y^2e^{-x} + y + e^x)} \\ &= \frac{1}{e^{-2x}y^2 + 1}\end{aligned}$$

Therefore,

$$\frac{ds}{dr} = \frac{1}{r^2 + 1}$$

$$s = \int \frac{1}{r^2 + 1} dr = \arctan(r)$$

we have

$$x = \arctan\left(\frac{y}{e^x}\right)$$

and

$$y = \tan(x)e^x$$

2.1.9 Infinitesimal Generator

The symmetry method can be used to solve first order differential equations. Many higher order differential equations can be reduced in order with the use of infinitesimal generators. For a one parameter Lie symmetry group $T_\alpha : (x, y) \mapsto (\bar{x}, \bar{y})$ there exist tangent vectors ξ and η. The infinitesimal generator for the symmetry is the partial differential operator

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$$

Example 2.11. [9] The following symmetry

$$(\bar{x}, \bar{y}) = (x, e^{\alpha x} y)$$

has an infinitesimal generator that has the tangent vectors:

$$\xi(x, y) = 0$$

and

$$\eta(x, y) = xye^{\alpha x}$$

So the infinitesimal generator for this symmetry is

$$X = xye^{\alpha x} \frac{\partial}{\partial y}$$

2.1.10 Infinitesimal Generator in Canonical Coordinates

Let $F(u, v)$ be an arbitrary smooth function. The infinitesimal generator X acts on F as

$$XF(u, v) = XF(u(x, y), v(x, y))$$

By the chain rule, we have

$$XF(u, v) = \xi(u_x F_u + v_x F_v) + \eta(u_y F_u + v_y F_v)$$

we get

$$\begin{aligned} XF(u,v) &= F_u(\xi(u_x+\eta u_y)) + F_v(\xi v_x + \eta v_y) \\ &= (Xu)F_u + (Xv)F_v \end{aligned}$$

but $F(u,v)$ is arbitrary function. Then the infinitesimal generator in the coordinates u and v are

$$X = (Xu)\frac{\partial}{\partial u} + (Xv)\frac{\partial}{\partial v}$$

If $(u,v) = (r,s)$ then

$$X = (Xr)\frac{\partial}{\partial r} + (Xs)\frac{\partial}{\partial s} \tag{2.1.51}$$

since $Xr = 0$ and $Xs = 1$, we get $X = \frac{\partial}{\partial s}$. The infinitesimal generator can be extended to equation with more variables. Suppose that $G(r,s)$ is a smooth function and let

$$F(x,y) = G(r(x,y), s(x,y)) \tag{2.1.52}$$

and therefore

$$F(\bar{x},\bar{y}) = G(\bar{r},\bar{s}) = G(r, s+\alpha)$$

Applying Taylor's theorem and (2.1.51), we get

$$F(\bar{x},\bar{y}) = G(r,s) + G_s(r,s)(s+\alpha-s) + \frac{G_{ss}(r,s)}{2!}(s+\alpha-s)^2 + \cdots = \sum_{j=0}^{\infty} \frac{\alpha^j}{j!}\frac{\partial^j}{\partial s^j}G(r,s)$$

since $X = \frac{\partial}{\partial s}$ in the canonical coordinates r and s, we can write

$$F(\bar{x},\bar{y}) = \sum_{j=0}^{\infty} \frac{\alpha^j}{j!} X^j G(r,s)$$

from (2.1.52), we get

$$F(\bar{x},\bar{y}) = \sum_{j=0}^{\infty} \frac{\alpha^j}{j!} X^j F(x,y) \tag{2.1.53}$$

Equation (2.1.53) can be written as

$$F(\bar{x},\bar{y}) = \sum_{j=0}^{\infty} e^{\alpha x} X^j F(x,y)$$

since the Taylor series expansion for e^x is $\sum_{j=0}^{\infty} \frac{x^j}{j!}$. This result can be generalized to any number of variables. Suppose that there are L variables, $z^1, \cdots, z^L$, and that the Lie symmetries are

$$\bar{z}^s(z^1,\cdots,z^L;\alpha) = z^s + \alpha\zeta^s(z^1,\cdots,z^L) + O(\alpha^2),\ \ s = 1,\cdots,L$$

where $\zeta^s = \frac{d\bar{z}^s}{d\alpha}|_{\alpha=0}$. The infinitesimal generator of the one-parameter Lie group is

$$X = \zeta^s(z^1,\cdots,z^L)\frac{\partial}{\partial z^s}$$

2.1.11 Lie Symmetries of Higher-Order Differential Equations

In this section, we want to describe the method for finding Lie symmetries of general ordinary differential equations. Consider kth-order ODE of the form

$$y^k = \omega(x,y,y',y'',\cdots,y^{(k-1)}),\quad y^{(p)} = \frac{d^p y}{dx^p} \tag{2.1.54}$$

where ω is locally smooth function. A symmetry of (2.1.54) is a diffeomorphism that maps the set of solutions of the ODE to itself. The action of T maps smooth curve to smooth curve.

$$T : (x,y,y',\cdots,y^{(k)}) \mapsto (\bar{x},\bar{y},\bar{y}',\cdots,\bar{y}^{(k)}) \tag{2.1.55}$$

where

$$\bar{y}^{(p)} = \frac{d^p\bar{y}}{d\bar{x}^p},\quad p = 1,\cdots,k \tag{2.1.56}$$

This mapping is called the kth prolongation of T. The function $\bar{y}^{(p)}$ can be solved recursively by using the chain rule

$$\bar{y}^{(p)} = \frac{d\bar{y}^{(p-1)}}{d\bar{x}} = \frac{D_x\bar{y}^{(p-1)}}{D_x\bar{x}},\quad \bar{y}^{(0)} \equiv \bar{y} \tag{2.1.57}$$

where D_x is the total derivative with respect to x defined by

$$D_x = \frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y} + \frac{d^2y}{dx^2}\frac{\partial}{\partial y'} + \cdots$$

So the symmetry condition for the ODE (2.1.54) is

$$\bar{y}^{(k)} = \omega(\bar{x}, \bar{y}, \bar{y}', \cdots, \bar{y}^{(k-1)}) \tag{2.1.58}$$

when (2.1.54) holds and where the functions $\bar{y}^{(p)}$ are defined by (2.1.56).
Since the symmetry condition (2.1.57) is nonlinear, Lie symmetry can be obtained by linearizing (2.1.57) about $\varepsilon = 0$.

Example 2.12. [9] The second-order ODE

$$y'' = 0, \quad x > 0 \tag{2.1.59}$$

has a symmetry

$$(\bar{x}, \bar{y}) = (\frac{1}{x}, \frac{y}{x}) \tag{2.1.60}$$

From (2.1.57), we get

$$\bar{y}'' = \frac{d\bar{y}'}{d\bar{x}} = \frac{D_x\bar{y}'}{D_x\bar{x}} \tag{2.1.61}$$

where

$$\bar{y}' = \frac{d\bar{y}}{d\bar{x}} = \frac{D_x\bar{y}}{D_x\bar{x}} = \frac{D_x(\frac{y}{x})}{D_x(\frac{1}{x})} \tag{2.1.62}$$

but

$$\begin{aligned} D_x(\frac{y}{x}) &= \frac{-y + y'x}{x^2} \\ D_x(\frac{1}{x}) &= \frac{-1}{x^2} \end{aligned} \tag{2.1.63}$$

Substituting into equation (2.1.61), we obtain

$$\bar{y}' = \frac{\frac{-y+y'x}{x^2}}{\frac{-1}{x^2}} = y - y'x \tag{2.1.64}$$

Substituting (2.1.64) into (2.1.61), we get

$$\begin{aligned}\bar{y}'' = \frac{D_x(y-y'x)}{D_x(\frac{1}{x})} &= \frac{\frac{\partial(y-y'x)}{\partial x} + y'\frac{\partial(y-y'x)}{y} + y''\frac{\partial(y-y'x)}{\partial y'}}{\partial_x \frac{1}{x}} \\ &= \frac{-y'+y'-y''x}{\frac{-1}{x^2}} = y''x^3\end{aligned}$$

so the symmetry condition is satisfied since $\bar{y}'' = 0$ when $y'' = 0$.
The general solution of the ODE is

$$y = c_1 x + c_2, \quad c_1, c_2 \in \mathbb{R}.$$

is mapped by transformation (2.1.60) to the solution

$$\bar{y} = \frac{y}{x} = \frac{c_1 x + c_2}{x} = c_1 + \frac{c_2}{x} = c_1 + c_2\bar{x}$$

So this symmetry acts on the set of solution curves by interchange the constant of integration c_1 and c_2.

The linearized symmetry condition for Lie symmetries is given by the same way as that for the first-order ODE using a Talyor series expansion. The prolonged Lie symmetry around $\alpha = 0$ is

$$\begin{aligned}\bar{x} &= x + \alpha\xi + O(\varepsilon^2) \\ \bar{y} &= y + \alpha\eta + O(\varepsilon^2) \\ \bar{y}^{(p)} &= y^{(p)} + \alpha\eta^{(p)} + O(\alpha^2), \quad p \geq 1\end{aligned} \tag{2.1.65}$$

where $\eta^{(p)}$ denotes the tangent vector that corresponds to the *pth* derivative of $\bar{y}$.
Substituting (2.1.65) into the symmetry condition (2.1.58), we get

$$\bar{y}^{(k)} = \omega(x + \alpha\xi + O(\alpha^2), y + \alpha\eta + O(\alpha^2), \cdots, y^{(k-1)} + \alpha\eta^{(k-1)} + O(\alpha^2))$$

Now apply Taylor's theorem about $\alpha = 0$, we get

$$= \omega(x, y, \cdots, y^{(k-1)}) + \alpha\left(\frac{\partial\omega}{\partial\bar{x}}\frac{\partial\bar{x}}{\partial\alpha}|_{\alpha=0} + \frac{\partial\omega}{\partial\bar{y}}\frac{\partial\bar{y}}{\partial\alpha}|_{\alpha=0} + \cdots + \frac{\partial\omega}{\partial\bar{y}^{(k-1)}}\frac{\partial\bar{y}^{(k-1)}}{\partial\alpha}|_{\alpha=0}\right) + O(\alpha^2)$$
$$= \omega(x, y, \cdots, y^{(k-1)}) + \alpha\left(\frac{\partial\omega}{\partial x}\xi + \frac{\partial\omega}{\partial y}\eta + \cdots + \frac{\partial\omega}{\partial y^{(k-1)}}\eta^{(k-1)}\right) + O(\alpha^2)$$
$$= \omega(x, y, \cdots, y^{(k-1)}) + \alpha\left(\omega_x\xi + \omega_y\eta + \cdots + \omega_{y^{(k-1)}}\eta^{(k-1)}\right) + O(\alpha^2) \tag{2.1.66}$$

when (2.1.54) holds, we have

$$\eta^{(k)} = \omega_x\xi + \omega_y\eta + \cdots + \omega_{y^{(k-1)}}\eta^{(k-1)} \tag{2.1.67}$$

which is called the linearized symmetry condition for kth-order ODE when (2.1.58) holds. The functional η^p can be solved recursively from (2.1.57), for $p = 1$ we get

$$\bar{y}' = \frac{D_x\bar{y}}{D_x\hat{x}} \tag{2.1.68}$$

Calculating $D_x\bar{y}$ by using the Taylor series expansion of $\bar{y}$:

$$\begin{aligned} D_x\bar{y} &= D_x(y + \alpha\eta + O(\alpha^2)) \\ &= \alpha\eta_x + y'(1 + \alpha\eta_y) + O(\alpha^2) \\ &= y' + \alpha(\eta_x + y'\eta_y) + O(\alpha^2) \\ &= y' + \alpha D_x\eta + O(\alpha^2) \end{aligned} \tag{2.1.69}$$

Calculating $D_x\bar{x}$ by using the Taylor series expansion of $\bar{x}$:

$$\begin{aligned} D_x\bar{x} &= D_x(x + \alpha\xi + O(\alpha^2)) \\ &= 1 + \alpha\xi_x + \alpha y'\xi_y + O(\alpha^2) \\ &= 1 + \alpha(\xi_x + y'\xi_y) + O(\alpha^2) \\ &= 1 + \alpha D_x\xi + O(\alpha^2) \end{aligned} \tag{2.1.70}$$

substituting (2.1.69) and (2.1.70) into (2.1.68), we have

$$\bar{y}' = \frac{y' + \alpha D_x\eta + O(\alpha^2)}{1 + \alpha D_x\xi + O(\alpha^2)} \tag{2.1.71}$$

multiplying the right hand side by

$$\frac{1-\alpha D_x\xi}{1-\alpha D_x\xi}$$

we have

$$\bar{y}' = \frac{(y'+\alpha D_x\eta)(1-\alpha D_x\xi)+O(\alpha^2)}{(1+\alpha D_x\xi)(1-\alpha D_x\xi)+O(\alpha^2)}$$

ignore terms of α^2 or higher so

$$\bar{y}' = y' + \alpha(D_x\eta - y'D_x\xi) + O(\alpha^2)$$

so

$$\eta^{(1)} = D_x\eta - y'D_x\xi \tag{2.1.72}$$

For $p = k-1$, calculating $D_x\bar{y}^{(k-1)}$ by using the Taylor series expansion of $\bar{y}^k$:

$$\begin{aligned} D_x\bar{y}^{(k-1)} &= D_x(y^{(k-1)} + \alpha\eta^{(k-1)} + O(\alpha^2)) \\ &= \alpha\eta_x^{(k-1)} + y'\alpha\eta_y^{(k-1)} + y''\alpha\eta_{y'}^{(k-1)} + \cdots + y^{(k)}(1+\alpha\eta_{y^{(k-1)}}^{(k-1)}) + O(\alpha^2) \\ &= y^{(k)} + \alpha\eta_x^{(k-1)} + y'\alpha\eta_y^{(k-1)} + y''\alpha\eta_{y'}^{(k-1)} + y^{(k)}\alpha\eta_{y^{(k-1)}}^{(k-1)} + O(\alpha^2) \\ &= y^{(k)} + \alpha D_x\eta^{(k-1)} + O(\alpha^2) \end{aligned} \tag{2.1.73}$$

Calculating $D_x\bar{x}$ by using the Taylor series expansion of $\bar{x}$:

$$\begin{aligned} D_x\bar{x} &= D_x(x + \alpha\xi + O(\alpha^2)) \\ &= 1 + \alpha\xi_x + y'\alpha\xi_y + O(\alpha^2) \\ &= 1 + \alpha(\xi_x + y'\xi_y) + O(\alpha^2) \\ &= 1 + \alpha D_x\xi + O(\alpha^2) \end{aligned} \tag{2.1.74}$$

substituting (2.1.73) and (2.1.74) into $\bar{y}^{(k)} = \frac{D_x\bar{y}^{(k)}}{D_x\bar{x}}$, we have

$$\bar{y}^{(k)} = \frac{y^{(k)} + \alpha D_x\eta^{(k-1)} + O(\alpha^2)}{1 + \alpha D_x\xi + O(\alpha^2)} \tag{2.1.75}$$

multiplying equation (2.1.75) by

$$\frac{1-\alpha D_x\xi}{1-\alpha D_x\xi}$$

we have

$$\bar{y}^{(k)} = \frac{(y^{(k)} + \alpha D_x \eta^{(k-1)})(1 - \alpha D_x \xi) + O(\alpha^2)}{(1 + \alpha D_x \xi)(1 - \alpha D_x \xi) + O(\alpha^2)}$$

ignore terms of α^2 or higher, we get

$$\bar{y}^{(k)} = y^{(k)} + \alpha(D_x \eta^{(k-1)} - y^{(k)} D_x \xi) + O(\alpha^2)$$

so

$$\eta^{(k)} = D_x \eta^{(k-1)} - y^{(k)} D_x \xi \tag{2.1.76}$$

To simplify (2.1.67), we introduce the prolonged infinitesimal generator

Definition 2.7. *[9] The infinitesimal generator $X^{(k)}$ is*

$$X^{(k)} = \xi \partial_x + \eta \partial_y + \eta^{(1)} \partial_{y'} + \cdots + \eta^{(k)} \partial_{y^{(k)}} \tag{2.1.77}$$

where $X^{(k)}$ is associated with the tangent vector in the space of variables $(x, y, y', \cdots, y^{(k)})$ and the coefficient of $\partial_{y^{(k)}}$ is the $O(\alpha^2)$ term in the expansion of $\bar{y}^{(k)}$.

So the Linearized symmetry for kth-order differential equations can be written as

$$X^{(k)}\left(y^{(k)} - \omega(x, y, y', \cdots, y^{(k-1)})\right) = 0, \textit{when } (2.1.54) \textit{ holds.}$$

Definition 2.8. *[9] A point transformation is a type of diffeomorphism of the form*

$$(\bar{x}, \bar{y}) = (\bar{x}(x, y), \bar{y}(x, y))$$

Moreover a point symmetry is any point transformation that is also a symmetry.

We restrict our attention to Lie symmetries for which ξ and η depend on x and y only which are called Lie point symmetries. To find the form of Lie point symmetries of an ODE (2.1.54) we find $\eta^{(p)}$, $p = 1, \cdots, k$ and then from (2.1.72) and (2.1.76), we have

$$\begin{aligned}\eta^{(1)} &= D_x\eta - y' D_x\xi \\ &= \eta_x + y'\eta_y - y'(\xi_x - \xi_y y') \\ &= \eta_x + y'(\eta_y - \xi_x) - \xi_y (y')^2\end{aligned} \tag{2.1.78}$$

To find $\eta^{(2)}$ we use $\eta^{(1)}$, we have

$$\begin{aligned}\eta^{(2)} &= D_x^{(1)}\eta - y'' D_x\xi \\ &= \eta_{xx} + y'\eta_{yx} - y'\xi_{xx} - \xi_{yx}(y')^2 + y'(\eta_{xy} + y'\eta_{yy} - y'\xi_{xy} - \xi_{yy}(y')^2 + y''(\eta_y - \xi_x \\ &- 2\xi_y y')) - y''(\xi_x + \xi_y y') \\ &= \eta_{xx} + y'(2\eta_{yx} - \xi_{xx}) + (y')^2(\eta_{yy} - 2\xi_{xy}) - \xi_{yy}(y')^3 + (\eta_y - 2\xi_x - 3\xi_y y')y''.\end{aligned} \tag{2.1.79}$$

The number of terms in η^k increases exponentially with k, we can use computer algebra to recommend for the study of high-order ODEs.
Substituting (2.1.78) and (2.1.79) into (**??**), we have

$$\begin{aligned}&\eta^{(2)} - \eta^{(1)}\omega_{y'} - \omega_x\xi - \omega_y\eta = 0 \\ &\eta_{xx} + y'(2\eta_{yx} - \xi_{xx}) + (y')^2(\eta_{yy} - 2\xi_{xy}) - \xi_{yy}(y')^3 + (\eta_y - 2\xi_x - 3\xi_y y')y'' - (\eta_x + y'(\eta_y \\ &- \xi_x) - \xi_y(y')^2)\omega_{y'} - \omega_x\xi - \omega_y\eta = 0\end{aligned} \tag{2.1.80}$$

Equation (2.1.80) is the linearized symmetry condition for second-order ODE. However, this equation is complicated to solve. It can be decomposed into a system of PDEs, which are the determining equations for the Lie point symmetries.

Example 2.13. [9] Consider the simplest second-order ODE

$$y'' = 0 \tag{2.1.81}$$

Since $y'' = 0$ so the linearized symmetry condition is

$$\eta_{xx} + y'(2\eta_{yx} - \xi_{xx}) + (y')^2(\eta_{yy} - 2\xi_{xy}) - \xi_{yy}(y')^3 = 0$$

This equation can be spilt into a system of four equations, called the determining equations

$$\eta_{xx} = 0 \tag{2.1.82}$$

$$2\eta_{yx} - \xi_{xx} = 0 \tag{2.1.83}$$

$$\eta_{yy} - 2\xi_{xy} = 0 \tag{2.1.84}$$

$$\xi_{yy} = 0 \tag{2.1.85}$$

Integrating Equation (2.1.85) with respect to y twice, we get

$$\xi(x, y) = A(x)y + B(x) \tag{2.1.86}$$

for arbitrary functions A and B.
Substitution (2.1.86) into (2.1.84)

$$\eta_{yy} = 2A'(x) \tag{2.1.87}$$

Integrating (2.1.87) with respect to y, we obtain

$$\eta_y = 2A'(x)y + C(x) \tag{2.1.88}$$

Integrating (2.1.87) with respect to y again, we obtain

$$\eta(x, y) = A'(x)y^2 + C(x)y + D(x)$$

Substituting $\xi(x, y)$ and $\eta(x, y)$ into equation (2.1.82) and (2.1.83), we get

$$\eta_{xx} = A'''(x)y^2 + C''(x)y + D''(x) = 0 \tag{2.1.89}$$

$$2\eta_{xy} - \xi_{xx} = 2(2A''(x)y + C'(x)) - (A''(x)y + B''(x)) \tag{2.1.90}$$

$$= 3A'(x)y + 2C'(x) - B'(x) = 0 \tag{2.1.91}$$

From equation (2.1.89) and (2.1.90), we obtain

$$A''(x) = 0$$
$$C''(x) = 0$$
$$D''(x) = 0$$
$$B''(x) = 2C'(x)$$

So the general solution for $A(x)$, $C(x)$ and $D(x)$ are

$$A(x) = c_1 x + c_2$$
$$C(x) = c_3 x + c_4$$
$$D(x) = c_5 x + c_6$$

where c_1, c_2, c_3, c_4, c_5 and c_6 are constants. The general solution for $B(x)$ is

$$B''(x) = 2C'(x) = 2c_3$$
$$B'(x) = 2c_3 x + c_7$$
$$B(x) = c_3 x^2 + c_7 x + c_8$$

Substituting $A(x)$, $B(x)$, $C(x)$ and $D(x)$ into $\xi(x, y)$ and $\eta(x, y)$, we get

$$\begin{aligned} \xi(x, y) &= A(x)y + B(x) \\ &= c_1 xy + c_2 y + c_3 x^2 + c_7 x + c_8 \end{aligned}$$

and

$$\begin{aligned} \eta(x, y) &= A'(x)y^2 + C(x)y + D(x) \\ &= c_1 y^2 + c_3 xy + c_4 y + c_5 x + c_6 \end{aligned}$$

So the infinitesimal generator for $y'' = 0$ is

$$X = (c_1 xy + c_2 y + c_3 x^2 + c_7 x + c_8)\partial_x + (c_1 y^2 + c_3 xy + c_4 y + c_5 x + c_6)\partial_y$$

3. SYMMETRY METHODS FOR DIFFERENCE EQUATIONS

In this chapter we extend the symmetry method for differential equations to non-linear difference equations. This method could be used to solve difference equations after adapting it to this field (see [7], [12]).
A transformation of a difference equation is a symmetry if every solution of the transformed equation is a solution of the original equation and vice versa.

Example 3.1. Let

$$T_\alpha : u_n \to \bar{u}_n = \alpha u_n, \quad for\ all\ \alpha \in \mathbb{R} - \{0\},$$

be a transformation on a linear homogeneous difference equation of order q

$$\alpha_q(n)u_{n+q} + \alpha_{q-1}(n)u_{n+q-1} + \cdots + \alpha_0(n)u_n = 0$$

Then T_α is a symmetry of the difference equation for all $\alpha \in \mathbb{R} - \{0\}$, since if $U_1(n), U_2(n), ..., U_q(n)$ are linearly independent solutions, then the general solution is

$$u_n = \sum_{i=1}^{q} c_i U_i(n).$$

The transformation T_α maps this solution to

$$\bar{u}_n = \alpha \sum_{i=1}^{q} c_i U_i(n) = \sum_{i=1}^{q} \bar{c}_i U_i(n), \quad \text{where} \quad \bar{c}_i = \alpha c_i.$$

for $i = 1, 2, \cdots, q$. So $\bar{u}_n$ is a solution of the original equation and vice versa. Thus, T_α is a symmetry for all $\alpha \in \mathbb{R} - \{0\}$.

Definition 3.1. *[6] Let*

$$\Gamma_\alpha : x \to \bar{x}(x;\alpha), \qquad \alpha \in (\alpha_0, \alpha_1),$$

where $\alpha_0 < 0$ *and* $\alpha_1 > 0$*, be a transformation, then it's a one parameter local Lie group if the following conditions are satisfied*

1. Γ_0 *is the identity map, that is,* $\bar{x} = x$ *when* $\alpha = 0$.
2. $\Gamma_\alpha \Gamma_\delta = \Gamma_{\alpha+\delta}$, $\forall \ \ \alpha, \ \delta$ *sufficiently close to* 0.
3. *Each* $\bar{x}$ *can be represented by a Taylor series in* α*, so*

$$\bar{x}(x;\alpha) = x + \alpha\xi(x) + O(\alpha^2).$$

Example 3.2. [7] Consider the difference equation:

$$u_{n+1} - u_n = 0. \tag{3.0.1}$$

and the transformation

$$T_\alpha : (n, u_n) \to (\bar{n}, \bar{u}_n) = (n, u_n + \alpha); \quad \alpha \in \mathbb{R} \tag{3.0.2}$$

T_α is a one parameter local Lie group, since

1. T_0 is the identity map since

$$T_0 : (n, u_n) \mapsto (\bar{n}, \bar{u}_n) = (n, u_n)$$

2. Let $T_\alpha : (n, u_n) \mapsto (n, u_n + \alpha)$ and $T_\delta : (n, u_n) \mapsto (n, u_n + \delta)$ be two transformations then

$$\begin{aligned} T_\alpha T_\delta &= T_\alpha(n, u_n + \delta) \\ &= (n, u_n + \delta + \alpha) \\ &= T_{\alpha+\delta} \end{aligned}$$

3. Each $\bar{u}_n$ can be represented as a Taylor series in α.

T_α is a symmetry for equation (3.0.1) since the solution of (3.0.1) is

$$u_n = c$$

and every transformation with $\alpha \neq 0$ maps each solution,

$$u_n = c \text{ } to \text{ } \bar{u}_n = c + \alpha$$

which can be written as $\bar{u}_n = \bar{c}$; $\bar{c} = c + \alpha$. So T_α is a Lie symmetry.

Note that n is a discrete variable that can't be changed by an arbitrarily small amount, so every one parameter local Lie group of symmetries must leave n unchanged. That is, $\bar{n} = n$ for all Lie symmetries of (3.0.1). The same argument applies to all difference equations.
We restrict our attention to Lie symmetries for which $\bar{u}_n$ depends on n and u_n only, which are called Lie point symmetries and take the form

$$\bar{n} = n, \qquad \bar{u}_n = u_n + \alpha Q(n, u_n) + O(\alpha^2), \tag{3.0.3}$$

where $Q(n, u_n)$ is a function of n and u_n that depends on the difference equation and is called a characteristic of the local Lie group.
If we replace n by $n + q$ in equation (3.0.3) we get

$$\bar{u}_{n+q} = u_{n+q} + \alpha Q(n + q, u_{n+q}) + O(\alpha^2),$$

which is called the prolongation formula for Lie point symmetries.

We want to invest symmetries and to use them to obtain exact solutions for difference equations.

Now consider the effect of changing variables from (n, u_n) to (n, s_n), and as (3.0.3) is a symmetry for each α sufficiently close to *zero*, we can apply Taylor's theorem about $\alpha = 0$, to obtain

$$\begin{aligned} \bar{s}_n &= s(\bar{n}, \bar{u}_n) \\ &= s(n, \bar{u}_n) \\ &= s(n, u_n + \alpha Q(n, u_n) + O(\alpha^2)) \end{aligned}$$

Now apply Taylor's theorem about $\alpha = 0$, we get

$$\begin{aligned}\bar{s}_n &= s(n,u_n) + \alpha(\frac{ds}{d\bar{u}_n})(\frac{d\bar{u}_n}{d\alpha}) \mid_{\alpha=0} + O(\alpha^2) \\ &= s(n,u_n) + \alpha s'(n,u_n)Q(n,u_n) + O(\alpha^2).\end{aligned}$$

If we denote the characteristic function with respect to (n, s_n) by $\bar{Q}(n, s_n)$ then we have:

$$\begin{aligned}\bar{s}_n &= s_n + \alpha\bar{Q}(n,s_n) + O(\alpha^2) \\ &= s(n,u_n) + \alpha s'(n,u_n)Q(n,u_n) + O(\alpha^2).\end{aligned}$$

So we get:

$$\bar{Q}(n,s_n) = s'(n,u_n)Q(n,u_n). \tag{3.0.4}$$

The coordinate s_n is called the canonical coordinate.

Example 3.3. [7] Consider changing the coordinates from (n, u_n) to (n, s_n), and symmetries for s_n,

$$(\bar{n}, \bar{s}_n) = (n, s_n + \alpha), \qquad \alpha \in \mathbb{R}.$$

Then the characteristic with respect to (n, s_n) is $\bar{Q}(n, s_n) = 1$, so by (3.0.4),

$$s'(n,u_n)Q(n,u_n) = 1,$$

which implies that

$$s(n,u_n) = \int \frac{du_n}{Q(n,u_n)} \tag{3.0.5}$$

Now, as an example if $Q(n, u_n) = u_n - 1$, then the canonical coordinate according to equation (3.0.5) is

$$s(n,u_n) = \int \frac{du_n}{u_n - 1} = \begin{cases} \ln(u_n - 1), & |u_n| > 1 \\ \ln(1 - u_n), & |u_n| < 1 \end{cases}$$

In this example, the map from u_n to s_n is not injective; it can't be inverted from s_n to u_n except if we specify whether $|u_n|$ is greater or less than 1.

3.1 Lie Symmetries of a Given First-Order Difference Equation

In this section we apply Lie symmetry to first order difference equations. This method depends on transforming the difference equation to a functional equation then we transform it to a differential equation.
Consider the first order difference equation of the form

$$u_{n+1} = w(n, u_n), \tag{3.1.1}$$

This equation can be solved using a one parameter local Lie group of symmetries.

For any transformation of a difference equation to be a symmetry, the set of solutions must be mapped to itself so the symmetry condition of equation (3.1.1) must be satisfied

$$\bar{u}_{n+1} = w(\bar{n}, \bar{u}_n) \quad \text{when} \quad u_{n+1} = w(n, u_n). \tag{3.1.2}$$

From the symmetry condition (3.1.2), we get

$$\begin{aligned} \bar{w}(n, u_n) &= w(\bar{n}, \bar{u}_n) \\ &= w(n, u_n + \alpha Q(n, u_n) + O(\alpha^2)) \\ &= w(n, u_n) + \alpha w'(n, u_n) Q(n, u_n) + O(\alpha^2). \end{aligned}$$

Also, we have

$$\bar{w}(n, u_n) = \bar{u}_{n+1} = u_{n+1} + \alpha Q(n+1, u_{n+1}) + O(\alpha^2).$$

So,

$$Q(n+1, u_{n+1}) = w'(n, u_n) Q(n, u_n). \tag{3.1.3}$$

This is called the linearized symmetry condition (LSC) for the given difference equation (3.1.1).
The linearized symmetry condition (3.1.3) is a linear functional equation that could be difficult to solve.

We can find the general solution of the linearized symmetry condition if we can solve the functional equation. But some functional equations can't be solved. However, there is no need to find the general solution of the linearized symmetry condition, as a single non-zero solution of this equation is sufficient to determine the general solution of the difference equation. For first order difference equations, a practical approach is to use an ansatz (trial solution) as a general solution of the linearized symmetry condition. Many physically important Lie point symmetries have characteristics of the form:

$$Q(n, u_n) = t_1(n)u_n^2 + t_2(n)u_n + t_3(n), \tag{3.1.4}$$

where $t_1(n)$, $t_2(n)$ and $t_3(n)$ are functions of n. By substituting (3.1.4) into the linearized symmetry condition (3.1.3) and comparing powers of u_n, we obtain the coefficients $t_1(n)$, $t_2(n)$ and $t_3(n)$.

Now, we know how to find a characteristic of first order difference equations, the remaining question is how can we use a characteristic to determine the general solution of the difference equation.
Consider the canonical coordinate (3.0.4), and as in example (3.3) let $\bar{Q}(n, u_n) = 1$, then

$$s_n = \int \frac{du_n}{Q(n, u_n)}.$$

To use a canonical coordinate to simplify or solve a given difference equation, firstly, we write the difference equation as a difference equation for s_n, then if we can solve this equation, it remains to write the solution in terms of the original variables. This happens only if we can invert the map from u_n to s_n . This condition is called compatibility condition, and s_n is called a compatible canonical coordinate.

Example 3.4. [7] Find the general solution

$$u_{n+1} = \frac{nu_n + 1}{u_n + n} = \omega(n, u_n), \qquad n \geq 2 \tag{3.1.5}$$

by using Lie point symmetry?

Solution.

$$\omega'(n, u_n) = \frac{\partial \omega(n, u_n)}{\partial u_n} = \frac{n^2 - 1}{(u_n + n)^2}$$

then the LSC for equation (3.1.5) is

$$Q\Big(n+1, \frac{nu_n+1}{u_n+n}\Big) = \frac{n^2-1}{(u_n+n)^2} Q(n,u_n)$$

with the ansatz (3.1.4), we get

$$t_1(n+1)u_{n+1}^2 + t_2(n+1)u_{n+1} + t_3(n+1) = \frac{n^2-1}{(u_n+n)^2}\big(t_1(n)u_n^2 + t_2(n)u_n + t_3(n)\big)$$

but $u_{n+1} = \frac{nu_{n+1}}{u_n+n}$, so

$$t_1(n+1)\Big(\frac{nu_{n+1}}{u_n+n}\Big)^2 + t_2(n+1)\frac{nu_{n+1}}{u_n+n} + t_3(n+1) = \frac{n^2-1}{(u_n+n)^2}\big(t_1(n)u_n^2 + t_2(n)u_n + t_3(n)\big)$$

multiplying by $(u_n+n)^2$, we obtain

$$n^2t_1(n+1)u_n^2 + 2nt_1(n+1)u_n + t_1(n+1) + nt_2(n+1)u_n^2 + (n^2+1)t_2(n+1)u_n + nt_2(n+1) +$$
$$t_3(n+1)u_n^2 + 2nt_3(n+1)u_n + n^2t_3(n+1) = (n^2-1)t_1(n)u_n^2 + (n^2-1)t_2(n)u_n + (n^2-1)t_3(n)$$

By comparing the powers of u, we get a system of difference equations:

$$u^2 \text{ terms} \;:\; n^2t_1(n+1) + nt_2(n+1) + t_3(n+1) = (n^2-1)t_1(n) \tag{3.1.6}$$

$$u \text{ terms} \;:\; 2nt_1(n+1) + (n^2+1)t_2(n+1) + 2nt_3(n+1)u_n = (n^2-1)t_2(n) \tag{3.1.7}$$

$$\textit{other} \text{ terms} \;:\; t_1(n+1) + nt_2(n+1) + n^2t_3(n+1) = (n^2-1)t_3(n) \tag{3.1.8}$$

subtracting (3.1.8) from (3.1.6), we get

$$t_1(n+1) - t_3(n+1) = t_1(n) - t_3,$$

so

$$t_1(n) - t_3(n) = q_1, \qquad q_1 \text{ is a constant},$$

adding (3.1.8) to (3.1.6), we get

$$(n^2+1)t_1(n+1) + 2nt_2(n+1) + (n^2+1)t_3(n+1) = (n^2-1)(t_1(n) + t_3(n) \tag{3.1.9}$$

subtracting (3.1.7) from (3.1.9) and adding (3.1.7) to (3.1.9), we get respectively

$$t_1(n+1) - t_2(n+1) + t_3(n+1) = \frac{n+1}{n-1}\big(t_1(n) - t_2(n) + t_3(n)\big),$$

$$t_1(n+1) + t_2(n+1) + t_3(n+1) = \frac{n-1}{n+1}\big(t_1(n) + t_2(n) + t_3\big),$$

which implies

$$t_1(n) - t_2(n) + t_3(n) = \left(\prod_{i=2}^{n-1} \frac{i+1}{i-1}\right) q_2 = \frac{n(n-1)}{2} q_2, \qquad q_2 \text{ is a constant},$$

$$t_1(n) + t_2 + t_3(n) = \left(\prod_{i=2}^{n-1} \frac{i-1}{i+1}\right) q_3 = \frac{2}{n(n-1)} q_3, \qquad q_3 \text{ is a constant}.$$

We have a linear system of difference equations for the coefficients $t_1(n), t_2(n)$ and $t_3(n)$,

$$\begin{aligned} t_1(n) \qquad - t_3(n) &= q_1 \\ t_1(n) - t_2(n) + t_3(n) &= \frac{n(n-1)}{2} q_2 \\ t_1(n) + t_2(n) + t_3(n) &= \frac{2}{n(n-1)} q_3 \end{aligned}$$

solving the system for the coefficients α_n, β_n and $t_3(n)$, hence

$$\begin{aligned} t_1(n) &= \frac{1}{2} q_1 + \frac{n(n-1)}{8} q_2 + \frac{1}{2n(n-1)} q_3 \\ t_2(n) &= -\frac{n(n-1)}{4} q_2 + \frac{1}{n(n-1)} q_3, \\ t_3(n)_n &= -\frac{1}{2} q_1 + \frac{n(n-1)}{8} q_2 + \frac{1}{2n(n-1)} q_3, \end{aligned}$$

so the characteristic

$$\begin{aligned} Q(n, u_n) &= t_1(n) u_n^2 + t_2(n) u_n + t_3(n) \\ &= q_1\Big(\frac{1}{2} u_n^2 - \frac{1}{2}\Big) + q_2\Big(\frac{n(n-1)}{8} u_n^2 - \frac{n(n-1)}{4} u_n + \frac{n(n-1)}{8}\Big) \\ &\qquad + q_3\Big(\frac{1}{2n(n-1)} u_n^2 + \frac{1}{n(n-1)} u_n + \frac{1}{2n(n-1)}\Big) \\ &= \frac{1}{2} q_1 (u_n^2 - 1) + \frac{n(n-1)}{8} q_2 (u_n^2 - 2u_n + 1) + \frac{1}{2n(n-1)} q_3 (u_n^2 + 2u_n + 1) \end{aligned}$$

we suppose $q_1 = 2,\ q_2 = 0$ and $q_3 = 0$, $Q(n, u_n) = u_n^2 - 1$ for case of computation there is no canonical coordinate $u_n = \pm 1$, if $u_2 = \pm 1$ then $u_n = u_2$. The appropriate real-valued canonical coordinate is

$$s_n = \int \frac{du_n}{u_n^2 - 1} = \begin{cases} \frac{1}{2} \ln \frac{u_n - 1}{u_n + 1}, & |u_n| > 1; \\ \frac{1}{2} \ln \frac{1 - u_n}{1 + u_n}, & |u_n| < 1, \end{cases}$$

but $u_2 \geq -1$ which implies $u_2 \in (-1, 1)$ or $(1, \infty)$ then u_n belong to the same interval, hence

$$s_n = \begin{cases} \frac{1}{2} \ln \frac{u_n - 1}{u_n + 1}, & u_n > 1; \\ \frac{1}{2} \ln \frac{1 - u_n}{1 + u_n}, & |u_n| < 1. \end{cases}$$

The transformation from u_n to s_n is not injective since $s_n(u_n) = s_n(\frac{1}{u_n})$, so s_n is not compatible canonical coordinate. To solve the difference equation and get u_n we seek an injective transformation to ensure the compatible condition.Therefore the problem of solving the difference equation splits into two separate parts.

Case 1: if $u_n > 1$, so

$$s_n = \frac{1}{2} \ln \frac{u_n - 1}{u_n + 1},$$

therefore the map from u_n to s_n is injective so the compatibility condition is satisfied and s_n is a compatible coordinate.

Now, consider the difference equation for s_n

$$\begin{aligned} s_{n+1} - s_n &= \frac{1}{2} \ln \Big(\frac{u_{n+1} - 1}{u_{n+1} + 1} \Big) - \frac{1}{2} \ln \Big(\frac{u_n - 1}{u_n + 1} \Big) \\ &= \frac{1}{2} \Big(\ln(u_{n+1} - 1) - \ln(u_{n+1} + 1) - \ln(u_n - 1) + \ln(u_n + 1) \Big) \\ &= \frac{1}{2} \Big(\ln \Big(\frac{nu_n + 1}{u_n + n} - 1 \Big) - \ln \Big(\frac{nu_n + 1}{u_n + n} + 1 \Big) - \ln(u_n - 1) + \ln(u_n + 1) \Big) \\ &= \frac{1}{2} \Big(\ln \Big(\frac{(u_n - 1)(n - 1)}{u_n + n} \Big) - \ln \Big(\frac{(u_n + 1)(n + 1)}{u_n + n} \Big) - \ln(u_n - 1) + \ln(u_n + 1) \Big) \\ &= \frac{1}{2} \ln \Big(\frac{n - 1}{n + 1} \Big), \end{aligned}$$

then

$$\begin{aligned} s_n &= s_2 + \frac{1}{2}\sum_{k=2}^{n-1} \ln\left(\frac{k-1}{k+1}\right) \\ &= \frac{1}{2}\ln\left(\frac{u_2-1}{u_2+1}\right) + \frac{1}{2}\ln\left(\prod_{k=2}^{n-1}\frac{k-1}{k+1}\right) \\ &= \frac{1}{2}\ln\left(\frac{u_2-1}{u_2+1}\right) + \frac{1}{2}\ln\left(\frac{2}{n(n-1)}\right) \\ &= \frac{1}{2}\ln\left(\frac{2(u_2-1)}{(u_2+1)n(n-1)}\right), \end{aligned}$$

so

$$\frac{1}{2}\ln\left(\frac{u_n-1}{u_n+1}\right) = \frac{1}{2}\ln\left(\frac{2(u_2-1)}{(u_2+1)n(n-1)}\right)$$

which implies

$$u_n = \frac{(u_2+1)n(n-1)+2(u_2-1)}{(u_2+1)n(n-1)-2(u_2-1)}.$$

case 2: if $|u_n| < 1$, so

$$s_n = \frac{1}{2}\ln\left(\frac{1-u_n}{1+u_n}\right),$$

therefore the map from u_n to s_n is injective so s_n is a compatible coordinate.

$$\begin{aligned} s_{n+1} - s_n &= \frac{1}{2}\ln\left(\frac{1-u_{n+1}}{1+u_{n+1}}\right) - \frac{1}{2}\ln\left(\frac{1-u_n}{1+u_n}\right) \\ &= \frac{1}{2}\ln\left(\frac{n-1}{n+1}\right), \end{aligned}$$

then

$$\begin{aligned} s_n &= s_2 + \frac{1}{2}\sum_{k=2}^{n-1} \ln\left(\frac{k-1}{k+1}\right) \\ &= \frac{1}{2}\ln\left(\frac{2(1-u_2)}{(1+u_2)n(n-1)}\right), \end{aligned}$$

so

$$\frac{1}{2}\ln\left(\frac{1-u_n}{1+u_n}\right) = \frac{1}{2}\ln\left(\frac{2(1-u_2)}{(1+u_2)n(n-1)}\right)$$

which implies

$$u_n = \frac{(u_2+1)n(n-1)+2(u_2-1)}{(u_2+1)n(n-1)-2(u_2-1)}.$$

thus, this value of u_n is valid for all $u_n \geq -1$. The general solution happens to include the solutions on which $Q(n, u_n) = 0$. ■

3.2 Lie Symmetries of a Given Higher-Order Difference Equation

Consider the ordinary difference equation of order q of the form

$$u_{n+q} = \omega(n, u_n, u_{n+1}, \cdots, u_{n+q-1}); \ \frac{\partial \omega}{\partial u_n} \neq 0 \tag{3.2.1}$$

where ω is locally smooth function.

Since for any transformation of a difference equation to be a symmetry, the set of solutions must be mapped to itself so the symmetry condition of equation (3.2.1) must be satisfied

$$\bar{u}_{n+q} = \omega(\bar{n}, \bar{u}_n, \bar{u}_{n+1}, \cdots, \bar{u}_{n+q-1}) \ \textit{when} \ (3.2.1) \ \textit{holds} \tag{3.2.2}$$

We restrict our attention to Lie symmetries of the form

$$\bar{n} = n, \quad \bar{u}_{n+q} = u_{n+q} + \alpha Q(n+q, u_{n+q}) + O(\alpha^2) \tag{3.2.3}$$

Substituting this equation into (3.2.2), we get

$$\begin{aligned}\omega(\bar{n}, \bar{u}_n, \bar{u}_{n+1}, \cdots, \bar{u}_{n+q-1}) = \omega(n, u_n + \alpha Q(n, u_n), u_{n+1} + \alpha Q(n+1, u_{n+1}), \cdots, \\ u_{n+q-1} + \alpha Q(n+q-1, u_{n+q-1}))\end{aligned}$$

Finding Taylor series of the right hand side about $\alpha = 0$, we get

$$\begin{aligned}\omega(\bar{n}, \bar{u}_n, \bar{u}_{n+1}, \cdots, \bar{u}_{n+q-1}) &= \omega(n, u_n, u_{n+1}, \cdots, u_{n+q-1}) + \alpha\Bigg(\frac{\partial \omega}{\partial \bar{u}_n}\frac{\partial \bar{u}_n}{\partial \alpha}\bigg|_{\alpha=0} + \frac{\partial \omega}{\partial \bar{u}_{n+1}}\frac{\partial \bar{u}_{n+1}}{\partial \alpha}\bigg|_{\alpha=0} \\ &\qquad + \cdots + \frac{\partial \omega}{\partial \bar{u}_{n+q-1}}\frac{\partial \bar{u}_{n+q-1}}{\partial \alpha}\bigg|_{\alpha=0}\Bigg) + O(\alpha^2) \\ &= \omega(n, u_n, u_{n+1}, \cdots, u_{n+q-1}) + \alpha\Bigg(\frac{\partial \omega}{\partial u_n}Q(n, u_n) + \frac{\partial \omega}{\partial u_{n+1}}Q(n+1, u_{n+1}) \\ &\qquad + \cdots + \frac{\partial \omega}{\partial u_{n+q-1}}Q(n+q-1, u_{n+q-1})\Bigg) + O(\alpha^2)\end{aligned} \tag{3.2.4}$$

also we have

$$\omega(\bar{n}, \bar{u}_n, \bar{u}_{n+1}, \cdots, \bar{u}_{n+q-1}) = \bar{u}_{n+q} = \omega(n, u_n, u_{n+1}, \cdots, u_{n+q-1}) + \alpha Q(n+q, u_{n+q}) + O(\alpha^2) \tag{3.2.5}$$

From equation (3.2.4) and (3.2.5), we get the linearized symmetry condition (LSC) for qth order difference equations

$$Q(n+q, u_{n+q}) = \frac{\partial \omega}{\partial u_n}Q(n, u_n) + \frac{\partial \omega}{\partial u_{n+1}}Q(n+1, u_{n+1}) + \cdots + \frac{\partial \omega}{\partial u_{n+q-1}}Q(n+q-1, u_{n+q-1})$$

To simplify this formula, we introduce the definition of the infinitesimal generator.

Definition 3.2. *The infinitesimal generator X is*

$$X = \sum_{k=0}^{q-1}(S^k Q(n, u_n))\frac{\partial}{\partial u_{n+k}}$$

where S^k is the forward shift operator defined as follows

$$S : n \mapsto n+1; \quad S^k u_n = u_{n+k}$$

and q is the order of the difference equation.

So the Linearized symmetry condition for qth order difference equations can be written as

$$S^k Q - X\omega = 0 \tag{3.2.6}$$

which is a linear functional equation for the characteristics $Q(n, u_n)$. However, functional equation are generally hard to solve. Lie symmetries are diffeomorphism, that is, $Q(n, u_n)$ is a smooth function, so the linearized symmetry condition can be solved by the method of differential elimination.
To transform equation (3.2.6) from a functional equation to a differential equation, we use the following steps:

- Firstly, in order to obtain an ODE for $Q(n, u_n)$ we apply appropriate differential operators to reduce the number of unknown functions then differentiate the LSC with respect to suitable independent variable and we may need to differentiate again.

- Secondly, from previous step we obtain an ordinary differential equation, which can be split by gathering together all terms with the same dependence and we solve it if possible, and obtain $Q(n, u_n)$. To find the coefficients of the terms of $Q(n, u_n)$, we plug it in the equations that we obtained in the previous steps which can be split into a system of linear difference equations by collecting all terms with the same dependence.

- After finding the characteristics $Q(n, u_n)$, we want to find the invariant v_n defined as

  Definition 3.3. *A function v_n is invariant under the Lie group of transformations T_α if $Xv_n = 0$, where $X = \sum_{k=0}^{q-1} S^k Q(n, u_n) \frac{\partial}{\partial u_{n+k}}$.*

  For equation (3.2.1), we suppose that the characteristic $Q(n, u_n)$ is known, then the invariant v_n can be found by solving the partial differential equation

  $$Xv_n = Q(n, u_n)\frac{\partial v_n}{\partial u_n} + SQ(n, u_n)\frac{\partial v_n}{\partial u_{n+1}} + \cdots + S^{q-1}Q(n, u_n)\frac{\partial v_n}{\partial u_{n+q-1}} = 0,$$

  and the general technique to solve the partial differential equations of this form is known as the method of characteristics and it is useful for finding analytic solutions.

To solve these equations, we use

$$\frac{du_n}{Q(n,u_n)} = \frac{du_{n+1}}{SQ(n,u_n)} = \cdots = \frac{du_{n+q-1}}{S^{q-1}Q(n,u_n)} := \frac{dv_n}{0}. \tag{3.2.7}$$

- We want to invest symmetries to reduce the order of difference equations. We find a compatible canonical coordinate s_n, which reduces the order by one.

- Finally, we obtain u_n from the canonical coordinate.

Example 3.5. [13] We investigate symmetries and solutions of the second-order difference equation

$$u_{n+2} = \frac{n + u_n u_{n+1}}{u_{n+1}} \tag{3.2.8}$$

where the initial values u_0 and u_1 are arbitrary nonzero real numbers. We want to find the solution of equation (3.2.8) by using Lie symmetries.

Solution. The linearized symmetry condition LSC to equation (3.2.8) is

$$Q(n+2,w) - \frac{\partial w}{\partial u_n}Q(n,u_n) - \frac{\partial w}{\partial u_{n+1}}Q(n+1,u_{n+1}) = 0,$$

but

$$\frac{\partial w}{\partial u_n} = 1,$$

and

$$\frac{\partial w}{\partial u_{n+1}} = \frac{-n}{u_{n+1}^2},$$

so the LSC is

$$Q(n+2,w) - Q(n,u_n) + \frac{n}{u_{n+1}^2}Q(n+1,u_{n+1}) = 0. \tag{3.2.9}$$

We apply the differential operator L to transform this functional equation to differential equation, given by

$$L = \frac{\partial}{\partial u_n} + \frac{u_{n+1}^2}{n}\frac{\partial}{\partial u_{n+1}},$$

to equation (3.2.9) to get

$$\frac{\partial}{\partial u_n}\left(Q(n+2,w)-Q(n,u_n)+\frac{n}{u_{n+1}^2}Q(n+1,u_{n+1})\right)+ \\ \left(\frac{u_{n+1}^2}{n}\frac{\partial}{\partial u_{n+1}}\right)\left(Q(n+2,w)-Q(n,u_n)+\frac{n}{u_{n+1}^2}Q(n+1,u_{n+1})\right)=0,$$

but

$$\frac{\partial}{\partial u_n}\left(Q(n+2,w)\right)=0,$$
$$\frac{\partial}{\partial u_n}\left(Q(n,u_n)\right)=Q'(n,u_n),$$
$$\frac{\partial}{\partial u_n}\left(\frac{n}{u_{n+1}^2}Q(n+1,u_{n+1})\right)=0,$$
$$\frac{\partial}{\partial u_{n+1}}\left(Q(n+2,w)\right)=0,$$
$$\frac{\partial}{\partial u_{n+1}}\left(Q(n,u_n)\right)=0,$$
$$\frac{\partial}{\partial u_{n+1}}\left(\frac{n}{u_{n+1}^2}Q(n+1,u_{n+1})\right)=\frac{n}{u_{n+1}^2}Q'(n+1,u_{n+1})+\frac{-2n}{u_{n+1}^3}Q(n+1,u_{n+1}),$$

this leads to

$$Q'(n+1,u_{n+1})-\frac{2}{u_{n+1}}Q(n+1,u_{n+1})-Q'(n,u_n)=0, \tag{3.2.10}$$

now, we differentiate this equation with respect to u_n keeping u_{n+1} fixed. As a result we obtain the ODE

$$-Q''(n,u_n)=0,$$

whose solution is given by

$$Q(n,u_n)=\alpha(n)u_n+\beta(n). \tag{3.2.11}$$

Next we substitute (3.2.11) into (3.2.10), we get

$$\alpha(n+1) - \frac{2}{u_{n+1}}(\alpha(n+1)u_{n+1} + \beta(n+1)) - \alpha(n) = 0,$$

the equation can be split by gathering together all terms with the same dependence upon u_{n+1}

$$-\alpha(n+1) - \alpha(n) - \frac{2}{u_{n+1}}\beta(n+1) = 0.$$

Now, we compare the two sides of the last equation, to obtain

$$-\alpha(n+1) - \alpha(n) = 0,$$

which is a first order linear difference equation whose general solution is

$$\alpha(n) = c(-1)^n,$$

where c is a constant. We have also

$$\beta(n+1) = 0 \quad \text{which implies} \quad \beta(n) = 0.$$

So

$$Q(n, u_n) = (-1)^n u_n.$$

We want to find the invariant using equation (3.2.7),

$$\frac{du_n}{(-1)^n u_n} = \frac{du_{n+1}}{(-1)^{n+1} u_{n+1}} = \frac{dv_n}{0},$$

Taking the first $\left(\frac{du_n}{(-1)^n u_n}\right)$ and second $\left(\frac{du_{n+1}}{(-1)^{n+1} u_{n+1}}\right)$ invariants, we get

$$\ln|u_n| + c^* = -\ln|u_{n+1}| \quad \text{which implies} \quad -c^* = \ln|u_{n+1}u_n|,$$

where $c^* \in \mathbb{R}$, so

$$t_1 = u_n u_{n+1} \quad \text{where} \quad t_1 = e^{-c^*},$$

also, we have

$$\frac{du_n}{u_n} = \frac{dv_n}{0},$$

which implies that

$$v_n = t, \quad \text{such that} \quad t = f(t_1),$$

where t_1 and t are constants.
We choose $f(t_1) = t_1$, therefore

$$v_n = u_n u_{n+1}. \tag{3.2.12}$$

Applying the shift operator to v_n yields

$$\begin{aligned} Sv_n = v_{n+1} &= u_{n+1}u_{n+2} \\ &= u_{n+1}\left(\frac{n + u_n u_{n+1}}{u_{n+1}}\right) \\ &= n + u_n u_{n+1} \\ &= n + v_n, \end{aligned} \tag{3.2.13}$$

So we have the equation

$$v_{n+1} - v_n = n,$$

which is a first order linear difference equation whose solution is given by

$$\begin{aligned} v_n &= v_0 + \sum_{k=0}^{n-1} k \\ &= v_0 + \frac{(n-1)n}{2}. \end{aligned} \tag{3.2.14}$$

Then by equations (3.2.12) and (3.2.14) we have

$$v_n = u_n u_{n+1} = v_0 + \frac{(n-1)n}{2},$$

Solving for u_{n+1} we obtain

$$u_{n+1} = \frac{v_0}{u_n} + \frac{(n-1)n}{2u_n}. \tag{3.2.15}$$

The order of Equation (3.2.8) has been reduced by one.
To solve equation (3.2.15) we need to obtain a canonical coordinate,

$$\begin{aligned} s_n &= \int \frac{du_n}{(-1)^n u_n} \\ &= (-1)^n \ln |u_n| . \end{aligned}$$

So $s_{n+1} - s_n$ is an invariant. Consequently,

$$\begin{aligned} s_{n+1} - s_n &= (-1)^{n+1} \ln|u_{n+1}| - (-1)^n \ln|u_n| \\ &= (-1)^{n+1} \ln|u_n u_{n+1}| \\ &= (-1)^{n+1} \ln|v_n| \\ &= (-1)^{n+1} \ln|*|\, v_0 + \frac{(n-1)n}{2}, \end{aligned} \tag{3.2.16}$$

The general solution of (3.2.16) is

$$\begin{aligned} s_n &= s_0 + \sum_{k=0}^{n-1} (-1)^{k+1} \ln|v_k| \\ &= \ln|u_0| + \sum_{k=0}^{n-1} (-1)^{k+1} \ln|*|\, u_0 u_1 + \frac{k(k-1)}{2}, \end{aligned}$$

but $s_n = (-1)^n \ln|u_n|$, so the general solution of (3.2.8)

$$\begin{aligned} u_n &= \exp\left((-1)^n \ln|u_0| + \sum_{k=0}^{n-1} (-1)^{k-n+1} \ln|*|\, u_0 u_1 + \frac{k(k-1)}{2} \right) \\ &= \exp\left((-1)^n \ln|u_0| \right) \cdot \exp\left(\sum_{k=0}^{n-1} (-1)^{k-n+1} \ln|*|\, u_0 u_1 + \frac{k(k-1)}{2} \right) \\ &= (u_0)^{(-1)^n} \prod_{k=0}^{n-1} \left(u_0 u_1 + \frac{k(k-1)}{2} \right)^{(-1)^{k-n+1}} \end{aligned}$$

■

4. SOLUTION AND BEHAVIOR OF A RATIONAL DIFFERENCE EQUATION

4.1 *Exact Solution of the Difference Equation* $u_{n+6} = \frac{u_n}{A_n + B_n u_n u_{n+3}}$

We investigate symmetries and solutions of the sixth-order difference equation

$$u_{n+6} = \frac{u_n}{A_n + B_n u_n u_{n+3}} := \omega \tag{4.1.1}$$

where the initial values $u_0, u_1, \cdots, u_5$ are arbitrary nonzero real numbers. We want to find the solution of equation (4.1.1) by using Lie symmetries.

The linearized symmetry condition (LSC) to equation (4.1.1) is

$$Q(n+6,\omega) - \frac{\partial\omega}{\partial u_n}Q(n,u_n) - \frac{\partial\omega}{\partial u_{n+1}}Q(n+1,u_{n+1}) - \frac{\partial\omega}{\partial u_{n+2}}Q(n+2,u_{n+2})$$
$$- \frac{\partial\omega}{\partial u_{n+3}}Q(n+3,u_{n+3}) - \frac{\partial\omega}{\partial u_{n+4}}Q(n+4,u_{n+4}) - \frac{\partial\omega}{\partial u_{n+5}}Q(n+5,u_{n+5}) = 0$$

but

$$\frac{\partial\omega}{\partial u_n} = \frac{A_n\omega^2}{u_n^2}$$
$$\frac{\partial\omega}{\partial u_{n+1}} = 0$$
$$\frac{\partial\omega}{\partial u_{n+2}} = 0$$
$$\frac{\partial\omega}{\partial u_{n+3}} = -B_n\omega^2$$

$$\frac{\partial\omega}{\partial u_{n+4}} = 0$$
$$\frac{\partial\omega}{\partial u_{n+5}} = 0$$

So the LSC is given by

$$Q(n+6,\omega) - \frac{A_n\omega^2}{u_n^2}Q(n,u_n) + B_n\omega^2 Q(n+3,u_{n+3}) = 0 \tag{4.1.2}$$

We apply the differential operator L to transform this functional equation to differential equation, given by

$$L = \frac{\partial}{\partial u_n} + \frac{\partial u_{n+3}}{\partial u_n}\frac{\partial}{\partial u_{n+3}}$$

where

$$\begin{aligned} \frac{\partial u_{n+3}}{\partial u_n} &= -\frac{\partial\omega/\partial u_n}{\partial\omega/\partial u_{n+3}} \\ &= \frac{A_n}{B_n u_n^2} \end{aligned}$$

so

$$L = \frac{\partial}{\partial u_n} + \frac{A_n}{B_n u_n^2}\frac{\partial}{\partial u_{n+3}}$$

to get

$$\begin{aligned} &\frac{\partial}{\partial u_n}\left(Q(n+6,\omega) - \frac{A_n\omega^2}{u_n^2}Q(n,u_n) + B_n\omega^2 Q(n+3,u_{n+3})\right) \\ &\quad + \frac{A_n}{B_n u_n^2}\frac{\partial}{\partial u_{n+3}}\left(Q(n+6,\omega) - \frac{A_n\omega^2}{u_n^2}Q(n,u_n) + B_n\omega^2 Q(n+3,u_{n+3})\right) = 0 \end{aligned}$$

but

$$\frac{\partial}{\partial u_n}(Q(n+6,\omega)) = 0$$
$$\frac{\partial}{\partial u_n}\left(\frac{A_n\omega^2}{u_n^2}Q(n,u_n)\right) = \frac{A_n\omega^2}{u_n^2}Q'(n,u_n) - \frac{2A_n\omega^2}{u_n^3}Q(n,u_n)$$

$$\frac{\partial}{\partial u_n}(B_n\omega^2 Q(n+3,u_{n+3}))=0$$
$$\frac{\partial}{\partial u_{n+3}}(Q(n+6,\omega))=0$$
$$\frac{\partial}{\partial u_{n+3}}\left(\frac{A_n\omega^2}{u_n^2}Q(n,u_n)\right)=0$$
$$\frac{\partial}{\partial u_{n+3}}(B_n\omega^2 Q(n+3,u_{n+3}))=B_n\omega^2 Q'(n+3,u_{n+3})$$

this leads to

$$-\frac{A_n\omega^2}{u_n^2}Q'(n,u_n)+\frac{2A_n\omega^2}{u_n^3}Q(n,u_n)+\frac{A_n\omega^2}{u_n^2}Q'(n+3,u_{n+3})=0$$

multiplying this equation by $-\frac{u_n^2}{A_n\omega^2}$, we get

$$Q'(n,u_n)-\frac{2}{u_n}Q(n,u_n)-Q'(n+3,u_{n+3})=0 \tag{4.1.3}$$

now, we differentiate equation (4.1.3) with respect to u_n keeping u_{n+3} fixed, we obtain the ODE

$$Q''(n,u_n)-\frac{2}{u_n}Q'(n,u_n)+\frac{2}{u_n^2}Q(n,u_n)=0$$

multiplying by u_n^2, we get

$$u_n^2Q''(n,u_n)-2u_nQ'(n,u_n)+2Q(n,u_n)=0$$

which is a Cauchy differential equation, whose solution is given by

$$Q(n,u_n)=\alpha_n u_n^2+\beta_n u_n \tag{4.1.4}$$

for some arbitrary functions α and β of n. We substitute equation (4.1.4) into (4.1.3), we get

$$-2\alpha_{n+3}u_{n+3}-(\beta_n+\beta_{n+3})=0$$

this equation can be spilt by gathering together all terms with the same dependence upon u_{n+3}

$$\begin{cases} 1:\beta_n+\beta_{n+3}=0 \\ u_{n+3}:2\alpha_{n+3}=0 \end{cases}$$

we have

$$\alpha_{n+3} = 0 \textit{ which implies } \alpha_n = 0$$

we have also

$$\beta_n + \beta_{n+3} = 0$$

which is a third order linear difference equation whose general solution is

$$\beta_n = (-1)^n c_1 + \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^n c_2 + \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^n c_3$$

for some arbitrary constants c_j, $j = 1, 2, 3$. So we get three characteristics and their corresponding generators as follows:

$$\begin{aligned} X_1 = {} & (-1)^n u_n \partial u_n + (-1)^{n+1} u_{n+1} \partial u_{n+1} + (-1)^{n+2} u_{n+2} \partial u_{n+2} + (-1)^{n+3} u_{n+3} \partial u_{n+3} \\ & + (-1)^{n+4} u_{n+4} \partial u_{n+4} + (-1)^{n+5} u_{n+5} \partial u_{n+5} \end{aligned}$$

$$\begin{aligned} X_2 = {} & \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^n u_n \partial u_n + \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^{n+1} u_{n+1} \partial u_{n+1} + \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^{n+2} u_{n+2} \partial u_{n+2} \\ & + \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^{n+3} u_{n+3} \partial u_{n+3} + \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^{n+4} u_{n+4} \partial u_{n+4} + \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^{n+5} u_{n+5} \partial u_{n+5} \end{aligned}$$

$$\begin{aligned} X_3 = {} & \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^n u_n \partial u_n + \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^{n+1} u_{n+1} \partial u_{n+1} + \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^{n+2} u_{n+2} \partial u_{n+2} \\ & + \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^{n+3} u_{n+3} \partial u_{n+3} + \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^{n+4} u_{n+4} \partial u_{n+4} + \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^{n+5} u_{n+5} \partial u_{n+5} \end{aligned}$$

Now, utilizing X_1, we introduce the canonical coordinate

$$S_n = \int \frac{du_n}{(-1)^n u_n} = (-1)^n \ln u_n$$

Now, we want to find the invariant using equation (3.2.7) we obtain

$$\frac{du_n}{(-1)^n u_n} = \frac{du_{n+1}}{(-1)^{n+1} u_{n+1}} = \cdots = \frac{du_{n+5}}{(-1)^{n+5} u_{n+5}} = \frac{dv_n}{0}$$

Taking the first $\left(\frac{du_n}{(-1)^n u_n}\right)$ and the second $\left(\frac{du_{n+1}}{(-1)^{n+1} u_{n+1}}\right)$ invariants, we get

$$\ln u_n + c_1 = -\ln u_{n+1} \textit{ which implies } -c_1 = \ln u_n u_{n+1}$$

where $c_1 \in \mathbb{R}$, so

$$T_1 = u_n u_{n+1}, \textit{ where } T_1 = e^{c_1}$$

Taking the first $\left(\frac{du_n}{(-1)^n u_n}\right)$ and the third $\left(\frac{du_{n+2}}{(-1)^{n+2} u_{n+2}}\right)$ invariants, we get

$$\ln u_n + c_2 = \ln u_{n+2} \textit{ which implies } c_2 = \ln \frac{u_{n+2}}{u_n}$$

where $c_2 \in \mathbb{R}$, so

$$T_2 = \frac{u_{n+2}}{u_n}, \textit{ where } T_2 = e^{c_2}$$

Taking the first $\left(\frac{du_n}{(-1)^n u_n}\right)$ and the fourth $\left(\frac{du_{n+3}}{(-1)^{n+3} u_{n+3}}\right)$ invariants, we get

$$\ln u_n + c_3 = -\ln u_{n+3} \textit{ which implies } -c_3 = \ln u_n u_{n+3}$$

where $c_3 \in \mathbb{R}$, so

$$T_3 = u_n u_{n+3}, \textit{ where } T_3 = e^{-c_3}$$

Taking the first $\left(\frac{du_n}{(-1)^n u_n}\right)$ and the fifth $\left(\frac{du_{n+4}}{(-1)^{n+4} u_{n+4}}\right)$ invariants, we get

$$\ln u_n + c_4 = \ln u_{n+4} \textit{ which implies } c_4 = \ln \frac{u_{n+4}}{u_n}$$

where $c_4 \in \mathbb{R}$, so

$$T_4 = \frac{u_{n+4}}{u_n}, \textit{ where } T_4 = e^{c_4}$$

Taking the first $\left(\frac{du_n}{(-1)^n u_n}\right)$ and the sixth $\left(\frac{du_{n+5}}{(-1)^{n+5} u_{n+5}}\right)$ invariants, we get

$$\ln u_n + c_5 = -\ln u_{n+5} \text{ which implies } -c_5 = \ln u_n u_{n+5}$$

where $c_5 \in \mathbb{R}$, so

$$T_5 = u_n u_{n+5}, \text{ where } T_5 = e^{-c_5}$$

Taking the second $\left(\frac{du_{n+1}}{(-1)^{n+1} u_{n+1}}\right)$ and the third $\left(\frac{du_{n+2}}{(-1)^{n+2} u_{n+2}}\right)$ invariants, we get

$$\ln u_{n+1} + c_6 = -\ln u_{n+2} \text{ which implies } -c_6 = \ln u_{n+1} u_{n+2}$$

where $c_6 \in \mathbb{R}$, so

$$T_6 = u_{n+1} u_{n+2}, \text{ where } T_6 = e^{-c_6}$$

Taking the second $\left(\frac{du_{n+1}}{(-1)^{n+1} u_{n+1}}\right)$ and the fourth $\left(\frac{du_{n+3}}{(-1)^{n+3} u_{n+3}}\right)$ invariants, we get

$$\ln u_{n+1} + c_7 = \ln u_{n+3} \text{ which implies } c_7 = \ln \frac{u_{n+3}}{u_{n+1}}$$

where $c_7 \in \mathbb{R}$, so

$$T_7 = u_{n+3} u_{n+1}, \text{ where } T_7 = e^{c_7}$$

Taking the second $\left(\frac{du_{n+1}}{(-1)^{n+1} u_{n+1}}\right)$ and the fifth $\left(\frac{du_{n+4}}{(-1)^{n+4} u_{n+4}}\right)$ invariants, we get

$$\ln u_{n+1} + c_8 = -\ln u_{n+4} \text{ which implies } -c_8 = \ln u_{n+1} u_{n+4}$$

where $c_8 \in \mathbb{R}$, so

$$T_8 = u_{n+1} u_{n+4}, \text{ where } T_8 = e^{-c_8}$$

Taking the second $\left(\frac{du_{n+1}}{(-1)^{n+1}u_{n+1}}\right)$ and the sixth $\left(\frac{du_{n+5}}{(-1)^{n+5}u_{n+5}}\right)$ invariants, we get

$$\ln u_{n+1} + c_9 = \ln u_{n+5} \textit{ which implies } c_9 = \ln \frac{u_{n+5}}{u_{n+1}}$$

where $c_9 \in \mathbb{R}$, so

$$T_9 = \frac{u_{n+5}}{u_{n+1}}, \textit{ where } T_9 = e^{c_9}$$

Taking the third $\left(\frac{du_{n+2}}{(-1)^{n+2}u_{n+2}}\right)$ and the fourth $\left(\frac{du_{n+3}}{(-1)^{n+3}u_{n+3}}\right)$ invariants, we get

$$\ln u_{n+2} + c_{10} = -\ln u_{n+3} \textit{ which implies } -c_{10} = \ln u_{n+2}u_{n+3}$$

where $c_{10} \in \mathbb{R}$, so

$$T_{10} = u_{n+2}u_{n+3}, \textit{ where } T_{10} = e^{-c_{10}}$$

Taking the third $\left(\frac{du_{n+2}}{(-1)^{n+2}u_{n+2}}\right)$ and the fifth $\left(\frac{du_{n+4}}{(-1)^{n+4}u_{n+4}}\right)$ invariants, we get

$$\ln u_{n+2} + c_{11} = \ln u_{n+4} \textit{ which implies } c_{11} = \ln \frac{u_{n+4}}{u_{n+2}}$$

where $c_{11} \in \mathbb{R}$, so

$$T_{11} = \frac{u_{n+4}}{u_{n+2}}, \textit{ where } T_{11} = e^{c_{11}}$$

Taking the third $\left(\frac{du_{n+2}}{(-1)^{n+2}u_{n+2}}\right)$ and the sixth $\left(\frac{du_{n+5}}{(-1)^{n+5}u_{n+5}}\right)$ invariants, we get

$$\ln u_{n+2} + c_{12} = -\ln u_{n+5} \textit{ which implies } -c_{12} = \ln u_{n+2}u_{n+5}$$

where $c_{12} \in \mathbb{R}$, so

$$T_{12} = u_{n+2}u_{n+5}, \textit{ where } T_{12} = e^{-c_{12}}$$

Taking the fourth $\left(\frac{du_{n+3}}{(-1)^{n+3}u_{n+3}}\right)$ and the fifth $\left(\frac{du_{n+4}}{(-1)^{n+4}u_{n+4}}\right)$ invariants, we get

$$\ln u_{n+3} + c_{13} = -\ln u_{n+4} \textit{ which implies } -c_{13} = \ln u_{n+3}u_{n+4}$$

where $c_{13} \in \mathbb{R}$, so

$$T_{13} = u_{n+3}u_{n+4}, \textit{ where } T_{13} = e^{-c_{13}}$$

Taking the fourth $\left(\frac{du_{n+3}}{(-1)^{n+3}u_{n+3}}\right)$ and the sixth $\left(\frac{du_{n+5}}{(-1)^{n+5}u_{n+5}}\right)$ invariants, we get

$$\ln u_{n+3} + c_{14} = \ln u_{n+5} \textit{ which implies } c_{14} = \ln \frac{u_{n+5}}{u_{n+3}}$$

where $c_{14} \in \mathbb{R}$, so

$$T_{14} = \frac{u_{n+5}}{u_{n+3}}, \textit{ where } T_{14} = e^{c_{14}}$$

Taking the fifth $\left(\frac{du_{n+4}}{(-1)^{n+4}u_{n+4}}\right)$ and the sixth $\left(\frac{du_{n+5}}{(-1)^{n+5}u_{n+5}}\right)$ invariants, we get

$$\ln u_{n+3} + c_{15} = -\ln u_{n+5} \textit{ which implies } -c_{15} = \ln u_{n+4}u_{n+5}$$

where $c_{15} \in \mathbb{R}$, so

$$T_{15} = u_{n+4}u_{n+5}, \textit{ where } T_{15} = e^{-c_{15}}$$

also, we have

$$\frac{du_n}{(-1)^n u_n} = \frac{dv_n}{0}$$

which implies that

$$v_n = T \textit{ such that } T = f(T_1, T_2, \cdots, T_{15})$$

where $T_i, i = 1, \cdots, 15$ and T are constants.

we choose $f(T_1, T_2, \cdots, T_{15}) = T_3 = u_n u_{n+3}$, therefore

$$v_n = u_n u_{n+3}$$

Apply the shift operator to v_n, yields

$$\begin{aligned} S^3 v_n = v_{n+3} &= u_{n+3} u_{n+6} \\ &= \frac{u_{n+3} u_n}{A_n + B_n u_n u_{n+3}} \\ &= \frac{v_n}{A_n + B_n v_n} \end{aligned}$$

which is a third order difference equation, which we solve to find v_n. From the above equation we have

$$\frac{1}{v_{n+3}} = A_n \frac{1}{v_n} + B_n$$

Let $\lambda_n = \frac{1}{v_n}$ we obtain

$$\lambda_{n+3} = A_n \lambda_n + B_n$$

This equation can be solved recursively. Let λ_0, λ_1 and λ_2 be given, then

$$\begin{aligned} \lambda_3 &= A_0 \lambda_0 + B_0 \\ \lambda_4 &= A_1 \lambda_1 + B_1 \\ \lambda_5 &= A_2 \lambda_2 + B_2 \\ \lambda_6 &= A_0 A_3 \lambda_0 + A_3 B_0 + B_3 \\ \lambda_7 &= A_1 A_4 \lambda_1 + A_4 B_1 + B_4 \\ \lambda_8 &= A_2 A_5 \lambda_2 + A_5 B_2 + B_5 \\ \lambda_9 &= A_0 A_3 A_6 \lambda_0 + A_3 A_6 B_0 + A_6 B_3 + B_6 \\ \lambda_{10} &= A_1 A_4 A_7 \lambda_1 + A_4 A_7 B_1 + A_7 B_4 + B_7 \\ \lambda_{11} &= A_2 A_5 A_8 \lambda_2 + A_5 A_8 B_2 + A_8 B_5 + B_8 \end{aligned}$$

So the general solution given by

$$\lambda_{3n+m} = \lambda_m \prod_{i_1=0}^{n-1} A_{3i_1+m} + \sum_{i_2=0}^{n-1} \left(B_{3i_2} \prod_{m=i_2+1}^{n-1} A_{3m} \right), \; m = 0, 1, 2$$

but $v_n = \frac{1}{\lambda n}$ so

$$\frac{1}{\lambda_{3n+m}} = \frac{1}{\lambda_m \prod_{i_1=0}^{n-1} A_{3i_1+m} + \sum_{i_2=0}^{n-1}\left(B_{3i_2} \prod_{m=i_2+1}^{n-1} A_{3m}\right)}, \ m = 0, 1, 2$$

we have

$$\begin{aligned} v_{3n+m} &= \frac{v_m}{\prod_{i_1=0}^{n-1} A_{3i_1+m} + v_m \sum_{i_2=0}^{n-1}\left(B_{3i_2} \prod_{m=i_2+1}^{n-1} A_{3m}\right)} \\ &= \frac{u_m u_{m+3}}{\prod_{i_1=0}^{n-1} A_{3i_1+m} + u_m u_{m+3} \sum_{i_2=0}^{n-1}\left(B_{3i_2} \prod_{m=i_2+1}^{n-1} A_{3m}\right)}, \ m = 0, 1, 2 \quad (4.1.5) \end{aligned}$$

The canonical coordinate s_n

$$s_n = \int \frac{du_n}{(-1)^n u_n} = (-1)^n \ln u_n$$

So $s_{n+3} - s_n$ is an invariant. Consequently,

$$\begin{aligned} s_{n+3} - s_n &= (-1)^{n+3} \ln u_{n+3} - (-1)^n \ln u_n \\ &= -(-1)^n (\ln u_{n+3} + \ln u_n) \\ &= (-1)^{n+1} (\ln u_{n+3} u_n) \\ &= (-1)^{n+1} (\ln v_n) \end{aligned}$$

whose general solution is given by

$$s_{3n+m} = c_0 + \left(\frac{-1}{2} + \frac{\sqrt{3}}{2} i\right)^{3n+m} c_1 + \left(\frac{-1}{2} - \frac{\sqrt{3}}{2} i\right)^{3n+m} c_2 + s_m + \sum_{i=0}^{n-1} (-1)^{i+1} \ln v_{3i+m}$$

where $m = 0, 1, 2$. From $s_n = (-1)^n \ln u_n$ we obtain

$$\begin{aligned} u_{3n+m} &= \exp\left((-1)^{3n+m} s_{3n+m}\right) \\ &= \exp\left((-1)^{3n+m} c_0 + (-1)^{3n+m}\left(\frac{-1}{2} + \frac{\sqrt{3}}{2} i\right)^{3n+m} c_1 + (-1)^{3n+m}\left(\frac{-1}{2} - \frac{\sqrt{3}}{2} i\right)^{3n+m} c_2 \right. \\ &\left. + (-1)^{3n+m} s_m + (-1)^{3n+m} \sum_{i=0}^{n-1} (-1)^{i+1} \ln v_{3i+m}\right), \ m = 0, 1, 2. \end{aligned}$$

To find c_0, c_1 and c_2 we substitute $n = 0$ and $m = 0, 1, 2$, we obtain

$$c_0 + c_1 + c_2 = 0$$
$$c_0 + \left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right)c_1 + \left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right)c_2 = 0$$
$$c_0 + \left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right)^2 c_1 + \left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right)^2 c_2 = 0$$

from this equations we get

$$c_0 = c_1 = c_2 = 0$$

So the general solution of equation

$$\begin{aligned} u_{3n+m} &= \exp\left((-1)^{3n+m} s_m + (-1)^{3n+m} \sum_{i=0}^{n-1} (-1)^{i+1} \ln v_{3i+m}\right) \\ &= u_m^{(-1)^n} \exp\left(\sum_{i=0}^{n-1} (-1)^{3n+m+i+1} \ln v_{3i+m}\right), \ m = 0, 1, 2 \\ &= u_m^{(-1)^n} \prod_{i=0}^{n-1} v_{3i+m}^{(-1)^{n+m+i+1}}, \ m = 0, 1, 2 \end{aligned} \tag{4.1.6}$$

Substituting $m = 0, 1, 2$ in equation (4.1.5), we obtain

$$\begin{aligned} v_{3n} &= \frac{u_0 u_3}{\prod_{i_1=0}^{n-1} A_{3i_1} + u_0 u_3 \sum_{i_2=0}^{n-1}\left(B_{3i_2} \prod_{m=i_2+1}^{n-1} A_{3m}\right)} \\ v_{3n+1} &= \frac{u_1 u_4}{\prod_{i_1=0}^{n-1} A_{3i_1+1} + u_1 u_4 \sum_{i_2=0}^{n-1}\left(B_{3i_2} \prod_{m=i_2+1}^{n-1} A_{3m}\right)} \\ v_{3n+2} &= \frac{u_2 u_5}{\prod_{i_1=0}^{n-1} A_{3i_1+2} + u_2 u_5 \sum_{i_2=0}^{n-1}\left(B_{3i_2} \prod_{m=i_2+1}^{n-1} A_{3m}\right)} \end{aligned} \tag{4.1.7}$$

Substituting $m = 0, 1, 2$ in equation (4.1.6), we get

$$u_{3n} = u_0^{(-1)^n} \prod_{i=0}^{n-1} v_{3i}^{(-1)^{n+i+1}}$$

$$u_{3n+1} = u_1^{(-1)^n} \prod_{i=0}^{n-1} v_{3i+1}^{(-1)^{n+i+2}}$$
$$u_{3n+2} = u_2^{(-1)^n} \prod_{i=0}^{n-1} v_{3i+2}^{(-1)^{n+i+3}}$$

Substituting $n = 1, 2, \cdots$, we obtain

$$u_3 = u_0^{-1} v_0$$
$$u_4 = u_1^{-1} v_1^{-1}$$
$$u_5 = u_2^{-1} v_2$$
$$u_6 = u_0 \frac{v_3}{v_0}$$
$$u_7 = u_1 \frac{v_1}{v_4}$$
$$u_8 = u_2 \frac{v_5}{v_2}$$
$$u_9 = u_3 \frac{v_6}{v_3}$$
$$u_{10} = u_4 \frac{v_4}{v_7}$$
$$u_{11} = u_5 \frac{v_8}{v_5}$$

So we can write the solution as

$$u_{6n} = u_0 \prod_{k=0}^{n-1} \frac{v_{6k+3}}{v_{6k}}$$
$$u_{6n+1} = u_1 \prod_{k=0}^{n-1} \frac{v_{6k+1}}{v_{6k+4}}$$
$$u_{6n+2} = u_2 \prod_{k=0}^{n-1} \frac{v_{6k+5}}{v_{6k+2}}$$
$$u_{6n+3} = u_3 \prod_{k=0}^{n-1} \frac{v_{6k+6}}{v_{6k+3}}$$

$$u_{6n+4} = u_4 \prod_{k=0}^{n-1} \frac{v_{6k+4}}{v_{6k+7}}$$

$$u_{6n+5} = u_5 \prod_{k=0}^{n-1} \frac{v_{6k+8}}{v_{6k+5}}$$

From equations (4.1.7), we get

$$u_{6n} = u_0 \prod_{k=0}^{n-1} \frac{\prod_{i_1=0}^{2k} A_{3i_1} + u_0 u_3 \sum_{i_2=0}^{2k} \left(B_{3i_2} \prod_{m=i_2+1}^{2k} A_{3m} \right)}{\prod_{i_1=0}^{2k-1} A_{3i_1} + u_0 u_3 \sum_{i_2=0}^{2k-1} \left(B_{3i_2} \prod_{m=i_2+1}^{2k-1} A_{3m} \right)}$$

$$u_{6n+1} = u_1 \prod_{k=0}^{n-1} \frac{\prod_{i_1=0}^{2k-1} A_{3i_1} + u_1 u_4 \sum_{i_2=0}^{2k-1} \left(B_{3i_2} \prod_{m=i_2+1}^{2k-1} A_{3m} \right)}{\prod_{i_1=0}^{2k} A_{3i_1} + u_1 u_4 \sum_{i_2=0}^{2k} \left(B_{3i_2} \prod_{m=i_2+1}^{2k} A_{3m} \right)}$$

$$u_{6n+2} = u_2 \prod_{k=0}^{n-1} \frac{\prod_{i_1=0}^{2k} A_{3i_1} + u_2 u_5 \sum_{i_2=0}^{2k} \left(B_{3i_2} \prod_{m=i_2+1}^{2k} A_{3m} \right)}{\prod_{i_1=0}^{2k-1} A_{3i_1} + u_2 u_5 \sum_{i_2=0}^{2k-1} \left(B_{3i_2} \prod_{m=i_2+1}^{2k-1} A_{3m} \right)}$$

$$u_{6n+3} = u_3 \prod_{k=0}^{n-1} \frac{\prod_{i_1=0}^{2k+1} A_{3i_1} + u_0 u_3 \sum_{i_2=0}^{2k+1} \left(B_{3i_2} \prod_{m=i_2+1}^{2k+1} A_{3m} \right)}{\prod_{i_1=0}^{2k} A_{3i_1} + u_0 u_3 \sum_{i_2=0}^{2k} \left(B_{3i_2} \prod_{m=i_2+1}^{2k} A_{3m} \right)}$$

$$u_{6n+4} = u_4 \prod_{k=0}^{n-1} \frac{\prod_{i_1=0}^{2k} A_{3i_1} + u_1 u_4 \sum_{i_2=0}^{2k} \left(B_{3i_2} \prod_{m=i_2+1}^{2k} A_{3m} \right)}{\prod_{i_1=0}^{2k+1} A_{3i_1} + u_1 u_4 \sum_{i_2=0}^{2k+1} \left(B_{3i_2} \prod_{m=i_2+1}^{2k+1} A_{3m} \right)}$$

$$u_{6n+5} = u_5 \prod_{k=0}^{n-1} \frac{\prod_{i_1=0}^{2k+1} A_{3i_1} + u_2 u_5 \sum_{i_2=0}^{2k+1} \left(B_{3i_2} \prod_{m=i_2+1}^{2k+1} A_{3m} \right)}{\prod_{i_1=0}^{2k} A_{3i_1} + u_2 u_5 \sum_{i_2=0}^{2k} \left(B_{3i_2} \prod_{m=i_2+1}^{2k} A_{3m} \right)}$$

4.2 Exact Solution Of The Difference Equation $u_{n+8} = \frac{u_n}{A_n + \alpha_n u_n u_{n+2} u_{n+4} u_{n+6}}$

We investigate symmetries and solutions of the of eighth-order difference equation

$$u_{n+8} = \frac{u_n}{A_n + B_n u_n u_{n+2} u_{n+4} u_{n+6}} := \omega \tag{4.2.1}$$

where the initial values $u_0, u_1, \cdots, u_7$ are arbitrary nonzero real numbers. We want to find the solution of equation (4.2.1) by using Lie symmetries.

The linearized symmetry condition (LSC) to equation (4.2.1) is

$$Q(n+8,\omega) - \frac{\partial\omega}{\partial u_n} Q(n,u_n) - \frac{\partial\omega}{\partial u_{n+1}} Q(n+1,u_{n+1}) - \frac{\partial\omega}{\partial u_{n+2}} Q(n+2,u_{n+2}) - \frac{\partial\omega}{\partial u_{n+3}} Q(n+3,u_{n+3})$$
$$- \frac{\partial\omega}{\partial u_{n+4}} Q(n+4,u_{n+4}) - \frac{\partial\omega}{\partial u_{n+5}} Q(n+5,u_{n+5}) - \frac{\partial\omega}{\partial u_{n+6}} Q(n+6,u_{n+6}) - \frac{\partial\omega}{\partial u_{n+7}} Q(n+7,u_{n+7}) = 0$$

but

$$\frac{\partial\omega}{\partial u_n} = \frac{A_n \omega^2}{u_n^2}$$
$$\frac{\partial\omega}{\partial u_{n+1}} = 0$$
$$\frac{\partial\omega}{\partial u_{n+2}} = -B_n \sigma^2 u_{n+4} u_{n+6}$$
$$\frac{\partial\omega}{\partial u_{n+3}} = 0$$
$$\frac{\partial\omega}{\partial u_{n+4}} = -B_n \sigma^2 u_{n+2} u_{n+6}$$
$$\frac{\partial\omega}{\partial u_{n+5}} = 0$$
$$\frac{\partial\omega}{\partial u_{n+6}} = -B_n \sigma^2 u_{n+2} u_{n+4}$$
$$\frac{\partial\omega}{\partial u_{n+7}} = 0$$

So the LSC is given by

$$Q(n+8,\omega)-\frac{A_n\omega^2}{u_n^2}Q(n,u_n)+B_n\omega^2u_{n+4}u_{n+6}Q(n+2,u_{n+2})+B_n\omega^2u_{n+2}u_{n+6}Q(n+4,u_{n+4})$$
$$+B_n\omega^2u_{n+2}u_{n+4}Q(n+6,u_{n+6})=0 \tag{4.2.2}$$

We apply the differential operator L to transform this functional equation to differential equation, given by

$$L=\frac{\partial}{\partial u_n}+\frac{\partial u_{n+2}}{\partial u_n}\frac{\partial}{\partial u_{n+2}}$$

where

$$\begin{aligned}\frac{\partial u_{n+2}}{\partial u_n} &= -\frac{\partial\omega/\partial u_n}{\partial\omega/\partial u_{n+2}}\\ &= \frac{A_n}{B_nu_n^2u_{n+4}u_{n+6}}\end{aligned}$$

so

$$L=\frac{\partial}{\partial u_n}+\frac{A_n}{B_nu_n^2u_{n+4}u_{n+6}}\frac{\partial}{\partial u_{n+3}}$$

to get

$$\frac{\partial}{\partial u_n}\Bigg(Q(n+8,\omega)-\frac{A_n\omega^2}{u_n^2}Q(n,u_n)+B_n\omega^2u_{n+4}u_{n+6}Q(n+2,u_{n+2})+B_n\omega^2u_{n+2}u_{n+6}$$
$$Q(n+4,u_{n+4})+B_n\omega^2u_{n+2}u_{n+4}Q(n+6,u_{n+6})\Bigg)+\frac{A_n}{B_nu_n^2u_{n+4}u_{n+6}}\frac{\partial}{\partial u_{n+2}}\Bigg(Q(n+8,\omega)$$
$$-\frac{A_n\omega^2}{u_n^2}Q(n,u_n)+B_n\omega^2u_{n+4}u_{n+6}Q(n+2,u_{n+2})+B_n\omega^2u_{n+2}u_{n+6}Q(n+4,u_{n+4})$$
$$+B_n\omega^2u_{n+2}u_{n+4}Q(n+6,u_{n+6})\Bigg)=0$$

but

$$\frac{\partial}{\partial u_n}(Q(n+8,\omega))=0$$
$$\frac{\partial}{\partial u_n}\left(\frac{A_n\omega^2}{u_n^2}Q(n,u_n)\right)=\frac{A_n\omega^2}{u_n^2}Q'(n,u_n)-\frac{2A_n\omega^2}{u_n^3}Q(n,u_n)$$
$$\frac{\partial}{\partial u_n}(B_n\omega^2u_{n+4}u_{n+6}Q(n+2,u_{n+2}))=0$$

$$\frac{\partial}{\partial u_n}(B_n\omega^2 u_{n+2}u_{n+6}Q(n+4,u_{n+4}))=0$$
$$\frac{\partial}{\partial u_n}(B_n\omega^2 u_{n+2}u_{n+4}Q(n+6,u_{n+6}))=0$$
$$\frac{\partial}{\partial u_{n+2}}(Q(n+8,\omega))=0$$
$$\frac{\partial}{\partial u_{n+2}}\left(\frac{A_n\omega^2}{u_n^2}Q(n,u_n)\right)=0$$
$$\frac{\partial}{\partial u_{n+2}}(B_n\omega^2 u_{n+4}u_{n+6}Q(n+2,u_{n+2}))=B_n\omega^2 u_{n+4}u_{n+6}Q'(n+2,u_{n+2})$$
$$\frac{\partial}{\partial u_{n+2}}(B_n\omega^2 u_{n+2}u_{n+6}Q(n+4,u_{n+4}))=B_n\omega^2 u_{n+6}Q(n+4,u_{n+4})$$
$$\frac{\partial}{\partial u_{n+2}}(B_n\omega^2 u_{n+2}u_{n+4}Q(n+6,u_{n+6}))=B_n\omega^2 u_{n+4}Q(n+6,u_{n+6})$$

this implies

$$-\frac{A_n\omega^2}{u_n}Q'(n,u_n)+\frac{2A_n\omega^2}{u_n^3}Q(n,u_n)+\frac{A_n\omega^2}{u_n^2}Q'(n+2,u_{n+2})+\frac{A_n\omega^2}{u_n^2u_{n+4}}Q(n+4,u_{n+4})$$
$$+\frac{A_n\omega^2}{u_n^2u_{n+6}}Q(n+6,u_{n+6})=0$$

multiplying this equation by $-\frac{u_n^2}{A_n\omega^2}$, we obtain

$$Q'(n.u_n)-\frac{2}{u_n}Q(n,u_n)-Q'(n+2,u_{n+2})-\frac{1}{u_{n+4}}Q(n+4,u_{n+4})-\frac{1}{u_{n+6}}Q(n+6,u_{n+6})=0 \tag{4.2.3}$$

now, we differentiate equation (4.2.3) with respect to u_n keeping u_{n+2}, u_{n+4} and u_{n+6} fixed, we get

$$Q''(n,u_n)-\frac{2}{u_n}Q'(n,u_n)+\frac{2}{u_n^2}Q(n,u_n)=0$$

multiplying by u_n^2, we get

$$u_n^2Q''(n,u_n)-2u_nQ'(n,u_n)+2Q(n,u_n)=0$$

which is Euler differential equation, whose solution is given by

$$Q(n,u_n)=\alpha_nu_n^2+\beta_nu_n \tag{4.2.4}$$

for some arbitrary function α and β of n. We substitute equation (4.2.4) into (4.2.3), we get

$$\beta_n + 2\alpha_{n+2}u_{n+2} + \beta_{n+2} + \alpha_{n+4}u_{n+4} + \beta_{n+4} + \alpha_{n+6}u_{n+6} + \beta_{n+6} = 0$$

$$\begin{cases} 1 : \beta_n + \beta_{n+2} + \beta_{n+4} + \beta_{n+6} = 0 \\ u_{n+2} : 2\alpha_{n+2} = 0 \\ u_{n+4} : \alpha_{n+4} = 0 \\ u_{n+6} : \alpha_{n+6} = 0 \end{cases}$$

we have

$$\alpha_{n+2}, \alpha_{n+4} \textit{ and } \alpha_{n+6} = 0 \textit{ which implies } \alpha_n = 0$$

we have also

$$\beta_n + \beta_{n+2} + \beta_{n+4} + \beta_{n+6} = 0$$

which is sixth order linear difference equation whose general solution is

$$\beta_n = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^n c_1 + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^n c_2 + \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^n c_3 + \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^n c_4 + (i)^n c_5 + (-i)^n c_6$$

for some arbitrary constants c_j, $j = 1, \cdots, 6$. So we get six characteristics and their corresponding generators are as follows:

$$X_1 = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^n u_n \partial u_n + (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^{n+1} u_{n+1} \partial u_{n+1} + (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^{n+2} u_{n+2} \partial u_{n+2} + (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^{n+3} u_{n+3} \partial u_{n+3} + (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^{n+4} u_{n+4} \partial u_{n+4} + (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^{n+5} u_{n+5} \partial u_{n+5} + (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^{n+6} u_{n+6} \partial u_{n+6} + (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^{n+7} u_{n+7} \partial u_{n+7}$$

$$X_2 = (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)^n u_n \partial u_n + (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)^{n+1} u_{n+1} \partial u_{n+1} + (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)^{n+2} u_{n+2} \partial u_{n+2} + (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)^{n+3} u_{n+3} \partial u_{n+3} + (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)^{n+4} u_{n+4} \partial u_{n+4} + (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)^{n+5} u_{n+5} \partial u_{n+5} + (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)^{n+6} u_{n+6} \partial u_{n+6} + (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)^{n+7} u_{n+7} \partial u_{n+7}$$

$$X_3 = (-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)^n u_n\partial u_n+(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)^{n+1}u_{n+1}\partial u_{n+1}+(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)^{n+2}u_{n+2}\partial u_{n+2}$$
$$+(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)^{n+3}u_{n+3}\partial u_{n+3}+(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)^{n+4}u_{n+4}\partial u_{n+4}+(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)^{n+5}u_{n+5}\partial u_{n+5}$$
$$+ (-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^{n+6}u_{n+6}\partial u_{n+6} + (-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^{n+7}u_{n+7}\partial u_{n+7}$$
$$X_4 = (-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i)^n u_n\partial u_n+(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i)^{n+1}u_{n+1}\partial u_{n+1}+(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i)^{n+2}u_{n+2}\partial u_{n+2}$$
$$+(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i)^{n+3}u_{n+3}\partial u_{n+3}+(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i)^{n+4}u_{n+4}\partial u_{n+4}+(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i)^{n+5}u_{n+5}\partial u_{n+5}$$
$$+ (-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)^{n+6}u_{n+6}\partial u_{n+6} + (-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)^{n+7}u_{n+7}\partial u_{n+7}$$
$$X_5 = (i)^n u_n\partial u_n+(i)^{n+1}u_{n+1}\partial u_{n+1}+(i)^{n+2}u_{n+2}\partial u_{n+2}+(i)^{n+3}u_{n+3}\partial u_{n+3}+(i)^{n+4}u_{n+4}$$
$$\partial u_{n+4} + (i)^{n+5}u_{n+5}\partial u_{n+5} + (i)^{n+6}u_{n+6}\partial u_{n+6} + (i)^{n+7}u_{n+7}\partial u_{n+7}$$
$$X_6 = (-i)^n u_n\partial u_n+(-i)^{n+1}u_{n+1}\partial u_{n+1}+(-i)^{n+2}u_{n+2}\partial u_{n+2}+(-i)^{n+3}u_{n+3}\partial u_{n+3}+(-i)^{n+4}$$
$$u_{n+4}\partial u_{n+4} + (-i)^{n+5}u_{n+5}\partial u_{n+5} + (-i)^{n+6}u_{n+6}\partial u_{n+6} + (-i)^{n+7}u_{n+7}\partial u_{n+7}$$

Now, utilizing X_1, we introduce the canonical coordinate

$$s_n = \int \frac{du_n}{(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^n u_n} = (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)^n \ln u_n$$

Now, we want to find the invariant using equation (3.2.7) we obtain

$$\frac{du_n}{(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^n u_n} = \frac{du_{n+1}}{(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^{n+1} u_{n+1}} = \cdots = \frac{du_{n+7}}{(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^{n+7} u_{n+7}} = \frac{dv_n}{0}$$

We proceed as in the previous equation to find the invariant. We get 28 constants $t_1, t_2, \cdots, t_{28}$. Using the first $\left(\frac{du_n}{(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)^n u_n}\right)$ and the fifth $\left(\frac{du_{n+4}}{(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)^{n+4} u_{n+4}}\right)$ invariants

$$\ln u_n + c_4 = -\ln u_{n+4} \textit{ which implies } -c_4 = \ln u_n u_{n+4}$$

where $c_4 \in \mathbb{R}$, so

$$t_4 = u_n u_{n+4}, \textit{ where } t_4 = e^{-c_4}$$

also, we have

$$\frac{du_n}{(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)^n u_n}=\frac{dv_n}{0}$$

which implies that

$$v_n = t \ such \ that \ t = f(t_1, t_2, \cdots, t_{28})$$

where $t_i, i = 1, \cdots, 28$ and t are constants.
we choose $f(t_1, t_2, \cdots, t_{28}) = t_4 = u_n u_{n+4}$, therefore

$$v_n = u_n u_{n+4}$$

Apply the shift operator to v_n, yields

$$\begin{aligned} Sv_n &= v_{n+1} = u_{n+1}u_{n+5} \\ S^2 v_n &= v_{n+2} = u_{n+2}u_{n+6} \\ S^3 v_n &= v_{n+3} = u_{n+3}u_{n+7} \\ S^4 v_n &= v_{n+4} = u_{n+4}u_{n+8} \\ &= \frac{u_{n+4}u_n}{A_n + B_n u_n u_{n+2} u_{n+4} u_{n+6}} \\ &= \frac{v_n}{A_n + B_n v_n v_{n+2}} \end{aligned}$$

So we have the equation:

$$\begin{aligned} v_{n+4} &= \frac{v_n}{A_n + B_n v_n v_{n+2}} \\ &= \theta(n, v_n, v_{n+1}, v_{n+2}, v_{n+3}) \end{aligned} \quad (4.2.5)$$

which is a fourth order difference equation, and we can use symmetry method to solve it. The (LSC) is:

$$\bar{Q}(n+4, v_{n+4}) - \frac{\partial\theta}{\partial v_n}\bar{Q}(n, v_n) - \frac{\partial\theta}{\partial v_{n+1}}\bar{Q}(n+1, v_{n+1}) - \frac{\partial\theta}{\partial v_{n+2}}\bar{Q}(n+2, v_{n+2}) \\ -\frac{\partial\theta}{\partial v_{n+3}}\bar{Q}(n+3, v_{n+3}) = 0$$

but

$$\frac{\partial\theta}{\partial v_n} = \frac{A_n\theta^2}{v_n^2}$$
$$\frac{\partial\theta}{\partial v_{n+1}} = 0$$
$$\frac{\partial\theta}{\partial v_{n+2}} = -B_n\theta^2$$
$$\frac{\partial\theta}{\partial v_{n+3}} = 0$$

So the LSC is given by

$$\bar{Q}(n+4, v_{n+4}) - \frac{A_n\theta^2}{v_n^2}\bar{Q}(n, v_n) + B_n\theta^2\bar{Q}(n+3, v_{n+3}) = 0 \tag{4.2.6}$$

We apply the differential operator L to transform this functional equation to differential equation, given by

$$L = \frac{\partial}{\partial v_n} + \frac{\partial v_{n+2}}{\partial v_n}\frac{\partial}{\partial v_{n+2}}$$

where

$$\begin{aligned}\frac{\partial v_{n+2}}{\partial v_n} &= -\frac{\partial\theta/\partial v_n}{\partial\theta/\partial v_{n+2}} \\ &= \frac{A_n}{B_n v_n^2}\end{aligned}$$

so

$$L = \frac{\partial}{\partial v_n} + \frac{A_n}{B_n v_n^2}\frac{\partial}{\partial v_{n+2}}$$

to get

$$\frac{\partial}{\partial v_n}\left(\bar{Q}(n+4,\theta) - \frac{A_n\theta^2}{v_n^2}\bar{Q}(n, v_n) + B_n\theta^2\bar{Q}(n+3, v_{n+3})\right) + \frac{A_n}{B_n v_n^2}\frac{\partial}{\partial v_{n+2}}\Bigg(\\ \bar{Q}(n+4,\theta) - \frac{A_n\theta^2}{v_n^2}\bar{Q}(n, v_n) + B_n\theta^2\bar{Q}(n+3, v_{n+3})\Bigg) = 0$$

but

$$\frac{\partial}{\partial v_n}\bar{Q}(n+4,\theta)=0$$
$$\frac{\partial}{\partial v_n}\left(\frac{A_n\theta^2}{v_n^2}\bar{Q}(n,v_n)\right)=\frac{A_n\theta^2}{v_n^2}\bar{Q}'(n,v_n)-\frac{2A_n\theta^2}{v_n^3}\bar{Q}(n,v_n)$$
$$\frac{\partial}{\partial v_n}(B_n\theta^2\bar{Q}(n+2,v_{n+2}))=0$$
$$\frac{\partial}{\partial v_{n+2}}\bar{Q}(n+4,\theta)=0$$
$$\frac{\partial}{\partial v_{n+2}}\left(\frac{A_n\theta^2}{v_n^2}\bar{Q}(n,v_n)\right)=0$$
$$\frac{\partial}{\partial v_{n+2}}(B_n\theta^2\bar{Q}(n+2,v_{n+2}))=B_n\theta^2\bar{Q}'(n+2,v_{n+2})$$

this implies

$$-\frac{A_n\theta^2}{v_n^2}\bar{Q}'(n,v_n)+\frac{2A_n\theta^2}{v_n^3}\bar{Q}(n,v_n)+\frac{A_n\theta^2}{v_n^2}\bar{Q}'(n+2,v_{n+2})=0$$

multiplying this equation by $-\frac{v_n^2}{A_n\theta^2}$, we get

$$\bar{Q}'(n,v_n)-\frac{2}{v_n}\bar{Q}(n,v_n)-\bar{Q}'(n+2,v_{n+2})=0 \qquad (4.2.7)$$

now, we differentiate equation (4.2.7) with respect to v_n keeping v_{n+2} fixed, we obtain the ODE

$$\bar{Q}''(n,v_n)-\frac{2}{v_n}\bar{Q}'(n,u_n)+\frac{2}{v_n^2}\bar{Q}(n,v_n)=0$$

multiplying by v_n^2, we get

$$v_n^2\bar{Q}''(n,v_n)-2v_n\bar{Q}'(n,v_n)+2\bar{Q}(n,v_n)=0$$

which is Euler differential equation, whose solution is given by

$$\bar{Q}(n,v_n)=\bar{\alpha}_n v_n^2+\bar{\beta}_n v_n \qquad (4.2.8)$$

for some arbitrary function $\bar{\alpha}$ and $\bar{\beta}$ of n. We substitute equation (4.2.8) into (4.2.7), we get

$$-2\bar{\alpha}_{n+2}v_{n+2}-(\bar{\beta}_n+\bar{\beta}_{n+2})=0$$

this equation can be spilt by gathering together all terms with the same dependence upon v_{n+2}

$$\begin{cases} 1 : \bar{\beta}_n + \bar{\beta}_{n+2} = 0 \\ v_{n+2} : 2\bar{\alpha}_{n+2} = 0 \end{cases}$$

we have

$$\bar{\alpha}_{n+2} = 0 \textit{ which implies } \bar{\alpha}_n = 0$$

we have also

$$\bar{\beta}_n + \bar{\beta}_{n+2} = 0$$

which is a second order linear difference equation whose general solution is

$$\bar{\beta}_n = (-i)^n c_1 + (i)^n c_2$$

for some arbitrary constants c_1 and c_2. So we get two characteristics and their corresponding generators are as follows:

$$X_1 = (i)^n v_n \partial v_n + (i)^{n+1} v_{n+1} \partial v_{n+1} + (i)^{n+2} v_{n+2} \partial v_{n+2} + (i)^{n+3} v_{n+3} \partial v_{n+3}$$
$$X_2 = (-i)^n v_n \partial v_n + (-i)^{n+1} v_{n+1} \partial v_{n+1} + (-i)^{n+2} v_{n+2} \partial v_{n+2} + (-i)^{n+3} v_{n+3} \partial v_{n+3}$$

Now, utilizing X_1, we introduce the canonical coordinate

$$\bar{s}_n = \int \frac{dv_n}{(i)^n v_n} = (-i)^n \ln v_n$$

Now, we want to find the invariant using equation (3.2.7) we obtain

$$\frac{dv_n}{(i)^n v_n} = \frac{dv_{n+1}}{(i)^{n+1} v_{n+1}} = \frac{dv_{n+2}}{(i)^{n+2} v_{n+2}} = \frac{dv_{n+3}}{(i)^{n+3} v_{n+3}} = \frac{d\bar{v}_n}{0}$$

We proceed as in the previous equation to find the invariant. We get 6 constants $z_1, z_2, \cdots, z_6$. Using the first $\left(\frac{dv_n}{(i)^n v_n}\right)$ and the third $\left(\frac{dv_{n+2}}{(i)^{n+2} v_{n+2}}\right)$ invariants, we get

$$\ln v_n + c_2 = -\ln v_{n+2} \textit{ which implies } -c_2 = \ln v_n v_{n+2}$$

where $c_2 \in \mathbb{R}$, so

$$z_2 = v_n v_{n+2},\ where\ z_2 = e^{-c_2}$$

also, we have

$$\frac{dv_n}{(i)^n v_n} = \frac{d\bar{v}_n}{0}$$

which implies that

$$\bar{v}_n = z\ such\ that\ z = f(z_1, z_2, \cdots, z_6)$$

where $z_i, i = 1, \cdots, 6$ and z are constants.
we choose $f(z_1, z_2, \cdots, z_6) = z_2 = v_n v_{n+2}$, therefore

$$\bar{v}_n = v_n v_{n+2}$$

Apply the shift operator to $\bar{v}_n$, yields

$$\begin{aligned} S\bar{v}_n &= \bar{v}_{n+1} = v_{n+1}v_{n+3} \\ S^2\bar{v}_n &= \bar{v}_{n+2} = u_{n+2}u_{n+4} \\ &= \frac{v_{n+2}v_n}{A_n + B_n v_n v_{n+2}} \\ &= \frac{\bar{v}_n}{A_n + B_n \bar{v}_n} \end{aligned}$$

which is a second order difference equation, which we solve to find $\bar{v}_n$. From the above equation we have

$$\frac{1}{\bar{v}_{n+2}} = A_n \frac{1}{\bar{v}_n} + B_n$$

Let $\gamma_n = \frac{1}{\bar{v}_n}$ we obtain

$$\gamma_{n+2} = A_n \gamma_n + B_n$$

This equation can be solved recursively. Let γ_0 and γ_1 be given, then

$$\begin{aligned}
\gamma_2 &= A_0\gamma_0 + B_0\\
\gamma_3 &= A_1\gamma_1 + B_1\\
\gamma_4 &= A_0A_2\gamma_0 + A_2B_0 + B_2\\
\gamma_5 &= A_1A_3\gamma_1 + A_3B_1 + B_3\\
\gamma_6 &= A_0A_2A_4\gamma_0 + A_2A_4B_0 + A_4B_2 + B_4\\
\gamma_7 &= A_1A_3A_5\gamma_1 + A_3A_5B_1 + A_5B_3 + B_5
\end{aligned}$$

So the general solution given by

$$\gamma_{2n+m} = \gamma_m \prod_{i_1=0}^{n-1} A_{2i_1+m} + \sum_{i_2=0}^{n-1}\left(B_{2i_2}\prod_{m=i_2+1}^{n-1} A_{2m}\right),\ m = 0, 1$$

but $\bar{v}_n = \frac{1}{\gamma n}$ so

$$\frac{1}{\gamma_{2n+m}} = \frac{1}{\gamma_m \prod_{i_1=0}^{n-1} A_{2i_1+m} + \sum_{i_2=0}^{n-1}\left(B_{2i_2}\prod_{m=i_2+1}^{n-1} A_{2m}\right)},\ m = 0, 1$$

we have

$$\bar{v}_{2n+m} = \frac{\bar{v}_m}{\prod_{i_1=0}^{n-1} A_{2i_1+m} + \bar{v}_m \sum_{i_2=0}^{n-1}\left(B_{2i_2}\prod_{m=i_2+1}^{n-1} A_{2m}\right)},\ m = 0, 1$$

The canonical coordinate $\bar{s}_n$

$$\bar{s}_n = \int \frac{dv_n}{(i)^n v_n} = (-i)^n \ln v_n$$

So $\bar{s}_{n+2} - \bar{s}_n$ is an invariant. Consequently,

$$\begin{aligned}
\bar{s}_{n+2} - \bar{s}_n &= (-i)^{n+2}\ln v_{n+2} - (-i)^n \ln v_n\\
&= -(-i)^n(\ln v_{n+2} + \ln v_n)\\
&= -(-i)^n(\ln v_{n+2}v_n)\\
&= (-1)^{n+1}(i)^n(\ln \bar{v}_n) \qquad (4.2.9)
\end{aligned}$$

we have

$$\bar{s}_2 = \bar{s}_0 - \ln \bar{v}_0$$
$$\bar{s}_3 = \bar{s}_1 + i \ln \bar{v}_1$$
$$\bar{s}_4 = \bar{s}_0 - \ln \bar{v}_0 + \ln \bar{v}_2$$
$$\bar{s}_5 = \bar{s}_1 + i \ln \bar{v}_1 - i \ln \bar{v}_3$$
$$\bar{s}_6 = \bar{s}_0 - \ln \bar{v}_0 + \ln \bar{v}_2 - \ln v_4$$
$$\bar{s}_7 = \bar{s}_1 + i \ln \bar{v}_1 - i \ln \bar{v}_3 + i \ln \bar{v}_5$$

So the general solution of the equation (4.2.9) is

$$\begin{aligned}\bar{s}_{2n+m} &= \bar{s}_m + \sum_{j=0}^{n-1} -(-i)^{2j+m} \ln \bar{v}_{2j+m}, \ m = 0, 1\\ &= \bar{s}_m + \sum_{j=0}^{n-1} -(-i)^{2j+m} \ln\left(\frac{\bar{v}_m}{\prod_{i_1=0}^{n-1} A_{2i_1+m} + \bar{v}_m \sum_{i_2=0}^{n-1}\left(B_{2i_2} \prod_{m=i_2+1}^{n-1} A_{2m}\right)}\right)\end{aligned}$$

where $m = 0, 1$. From $\bar{s}_n = (-i)^n \ln v_n$ we obtain

$$\begin{aligned}v_{2n+m} &= \exp\left((i)^{2n+m} \bar{s}_{2n+m}\right)\\ &= \exp\left((i)^{2n+m} \bar{s}_m + (i)^{2n+m} \sum_{j=0}^{n-1} -(-i)^{2j+m} \ln \bar{v}_{2j+m}\right), \ m = 0, 1\\ &= v_m^{(-1)^n} \prod_{j=0}^{n-1}\left(\frac{\bar{v}_m}{\prod_{i_1=0}^{n-1} A_{2i_1+m} + \bar{v}_m \sum_{i_2=0}^{n-1}\left(B_{2i_2} \prod_{m=i_2+1}^{n-1} A_{2m}\right)}\right)^{(-1)^{n+j+1}}\end{aligned}$$

$$= (u_m u_{m+4})^{(-1)^n} \prod_{j=0}^{n-1}\left(\frac{u_m u_{m+2} u_{m+4} u_{m+6}}{\prod_{i_1=0}^{n-1} A_{2i_1+m} + u_m u_{m+2} u_{m+4} u_{m+6} \sum_{i_2=0}^{n-1}\left(B_{2i_2} \prod_{m=i_2+1}^{n-1} A_{2m}\right)}\right)^{(-1)^{n+j+1}} \tag{4.2.10}$$

The canonical coordinate s_n is

$$s_n = \int \frac{du_n}{\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^n u_n} = \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^n \ln u_n$$

So $s_{n+4} - s_n$ is an invariant. Consequently,

$$\begin{aligned} s_{n+4} - s_n &= \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^{n+4} \ln u_n - \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^n \ln u_n \\ &= -\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^n (\ln u_{n+4} + \ln u_n) \\ &= -\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^n \ln v_n \end{aligned}$$

we have

$$\begin{aligned} s_4 &= s_0 - \ln v_0 \\ s_5 &= s_1 - \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \ln v_1 \\ s_6 &= s_2 + i \ln v_2 \end{aligned}$$

$$\begin{aligned} s_7 &= s_3 + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \ln v_3 \\ s_8 &= s_0 - \ln v_0 + \ln v_4 \\ s_9 &= s_1 - \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \ln v_1 - \left(\frac{\sqrt{2}(1-i)^5}{8}\right) \ln v_5 \\ s_{10} &= s_2 + i \ln v_2 - i \ln v_6 \\ s_7 &= s_3 + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \ln v_3 + \left(\frac{\sqrt{2}(1-i)^7}{16} \ln v_7\right) \end{aligned}$$

whose general solution is given by

$$\begin{aligned} s_{4n+m} = c_0 + (-1)^{4n+m} c_1 + (i)^{4n+m} c_2 + (-i)^{4n+m} c_3 + s_m \\ - \sum_{j=1}^{n-1} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^{4j+m} \ln v_{4j+m};\ m = 0, 1, 2, 3 \end{aligned}$$

Also, we have $u_n = \exp((\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^n s_n)$, , from this we get

$$u_{4n+m} = \exp\Bigg((\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^{4n+m}c_0 + (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^{4n+m}(-1)^{4n+m}c_1 + (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^{4n+m}$$
$$(i)^{4n+m}c_2 + (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^{4n+m}(-i)^{4n+m}c_3 + (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^{4n+m}s_m + (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^{4n+m}$$
$$\sum_{j=1}^{n-1} -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^{4j+m} \ln v_{4j+m}\Bigg); \ m = 0, 1, 2, 3 \tag{4.2.11}$$

To find c_0, c_1, c_2 and c_3, we substitute $n = 0$ and $m = 0, 1, 2, 3$, we obtain

$$c_0 + c_1 + c_2 + c_3 = 0$$
$$c_0 - c_1 + ic_2 - ic_3 = 0$$
$$c_0 + c_1 - c_2 - c_3 = 0$$
$$c_0 - c_1 - ic_2 + ic_3 = 0$$

from this equations we get

$$c_0 = c_1 = c_2 = c_3 = 0$$

So the greneral solution of equation (4.2.11) is

$$u_{4n+m} = \exp(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^{4n+m}s_m + (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)^{4n+m} \sum_{j=1}^{n-1}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^{4j+m} \ln v_{4j+m}$$
$$= u_m^{(-1)^n} \prod_{j=0}^{n-1}(v_{4j+m})^{(-1)^{n+j+1}}; \ m = 0, 1, 2, 3 \tag{4.2.12}$$

Substituting $m = 0, 1$ in equation (4.2.10), we obtain

$$v_{2n} = (u_0u_4)^{(-1)^n} \prod_{j=0}^{n-1}\left(\frac{u_0u_2u_4u_6}{\prod_{i_1=0}^{n-1} A_{2i_1+m} + u_0u_2u_4u_6 \sum_{i_2=0}^{n-1}\left(B_{2i_2}\prod_{m=i_2+1}^{n-1} A_{2m}\right)}\right)^{(-1)^{n+j+1}}$$
$$v_{2n+1} = (u_1u_5)^{(-1)^n} \prod_{j=0}^{n-1}\left(\frac{u_1u_3u_5u_7}{\prod_{i_1=0}^{n-1} A_{2i_1+m} + u_1u_3u_5u_7 \sum_{i_2=0}^{n-1}\left(B_{2i_2}\prod_{m=i_2+1}^{n-1} A_{2m}\right)}\right)^{(-1)^{n+j+1}} \tag{4.2.13}$$

Substituting $m = 0, 1, 2, 3$ in equation (4.2.12), we get

$$u_{4n} = u_m^{(-1)^0} \prod_{j=0}^{n-1} (v_{4j})^{(-1)^{n+j+1}}$$

$$u_{4n+1} = u_m^{(-1)^1} \prod_{j=0}^{n-1} (v_{4j+1})^{(-1)^{n+j+1}}$$

$$u_{4n+2} = u_m^{(-1)^2} \prod_{j=0}^{n-1} (v_{4j+2})^{(-1)^{n+j+1}}$$

$$u_{4n+3} = u_m^{(-1)^3} \prod_{j=0}^{n-1} (v_{4j+3})^{(-1)^{n+j+1}}$$

Substituting $n = 1, 2, \cdots$, we obtain

$$u_4 = u_0^{-1} v_0$$

$$u_5 = u_1^{-1} v_1$$

$$u_6 = u_2^{-1} v_2$$

$$u_7 = u_3^{-1} v_3$$

$$u_8 = u_0 \frac{v_4}{v_0}$$

$$u_9 = u_1 \frac{v_5}{v_1}$$

$$u_{10} = u_2 \frac{v_6}{v_2}$$

$$u_{11} = u_3 \frac{v_7}{v_3}$$

$$u_{12} = u_4 \frac{v_8}{v_4}$$

$$u_{13} = u_5 \frac{v_9}{v_5}$$

$$u_{14} = u_6 \frac{v_{10}}{v_6}$$

$$u_{15} = u_7 \frac{v_{11}}{v_7}$$

So we can write the soulution as

$$u_{8n} = u_0 \prod_{k=0}^{n-1} \frac{v_{8k+4}}{v_{8k}}$$

$$u_{8n+1} = u_1 \prod_{k=0}^{n-1} \frac{v_{8k+5}}{v_{6k+1}}$$

$$u_{8n+2} = u_2 \prod_{k=0}^{n-1} \frac{v_{8k+6}}{v_{8k+2}}$$

$$u_{8n+3} = u_3 \prod_{k=0}^{n-1} \frac{v_{8k+7}}{v_{8k+3}}$$

$$u_{8n+4} = u_4 \prod_{k=0}^{n-1} \frac{v_{8k+8}}{v_{8k+4}}$$

$$u_{8n+5} = u_5 \prod_{k=0}^{n-1} \frac{v_{8k+9}}{v_{8k+5}}$$

$$u_{8n+6} = u_6 \prod_{k=0}^{n-1} \frac{v_{8k+10}}{v_{8k+6}}$$

$$u_{8n+7} = u_7 \prod_{k=0}^{n-1} \frac{v_{8k+11}}{v_{8k+7}}$$

From equtions (4.2.13), we get

$$u_{8n} = u_0 \prod_{k=0}^{n-1} \frac{\prod_{i_1=0}^{4k+1} A_{2i_1} + u_0u_2u_4u_6 \sum_{i_2=0}^{4k+1}\left(B_{2i_2} \prod_{m=i_2+1}^{4k+1} A_{2m}\right)}{\prod_{i_1=0}^{4k-1} A_{2i_1} + u_0u_2u_4u_6 \sum_{i_2=0}^{4k-1}\left(B_{2i_2} \prod_{m=i_2+1}^{4k-1} A_{2m}\right)}$$

$$u_{8n+1} = u_1 \prod_{k=0}^{n-1} \frac{\prod_{i_1=0}^{4k+1} A_{2i_1+1} + u_1u_3u_5u_7 \sum_{i_2=0}^{4k+1}\left(B_{2i_2} \prod_{m=i_2+1}^{4k+1} A_{2m}\right)}{\prod_{i_1=0}^{4k-1} A_{2i_1+1} + u_1u_3u_5u_7 \sum_{i_2=0}^{4k-1}\left(B_{2i_2} \prod_{m=i_2+1}^{4k-1} A_{2m}\right)}$$

$$u_{8n+2} = u_2 \prod_{k=0}^{n-1} \frac{\prod_{i_1=0}^{4k+2} A_{2i_1} + u_0u_2u_4u_6 \sum_{i_2=0}^{4k+2}\left(B_{2i_2} \prod_{m=i_2+1}^{4k+2} A_{2m}\right)}{\prod_{i_1=0}^{4k} A_{2i_1} + u_0u_2u_4u_6 \sum_{i_2=0}^{4k}\left(B_{2i_2} \prod_{m=i_2+1}^{4k} A_{2m}\right)}$$

$$u_{8n+3} = u_3 \prod_{k=0}^{n-1} \frac{\prod_{i_1=0}^{4k+2} A_{2i_1+1} + u_1u_3u_5u_7 \sum_{i_2=0}^{4k+2} \left(B_{2i_2} \prod_{m=i_2+1}^{4k+2} A_{2m} \right)}{\prod_{i_1=0}^{4k} A_{2i_1+1} + u_1u_3u_5u_7 \sum_{i_2=0}^{4k} \left(B_{2i_2} \prod_{m=i_2+1}^{4k} A_{2m} \right)}$$

$$u_{8n+4} = u_4 \prod_{k=0}^{n-1} \frac{\prod_{i_1=0}^{4k+3} A_{2i_1} + u_0u_2u_4u_6 \sum_{i_2=0}^{4k+3} \left(B_{2i_2} \prod_{m=i_2+1}^{4k+3} A_{2m} \right)}{\prod_{i_1=0}^{4k+1} A_{2i_1} + u_0u_2u_4u_6 \sum_{i_2=0}^{4k+1} \left(B_{2i_2} \prod_{m=i_2+1}^{4k+1} A_{2m} \right)}$$

$$u_{8n+5} = u_5 \prod_{k=0}^{n-1} \frac{\prod_{i_1=0}^{4k+3} A_{2i_1+1} + u_1u_3u_5u_7 \sum_{i_2=0}^{4k+3} \left(B_{2i_2} \prod_{m=i_2+1}^{4k+3} A_{2m} \right)}{\prod_{i_1=0}^{4k+1} A_{2i_1+1} + u_1u_3u_5u_7 \sum_{i_2=0}^{4k+1} \left(B_{2i_2} \prod_{m=i_2+1}^{4k+1} A_{2m} \right)}$$

$$u_{8n+6} = u_6 \prod_{k=0}^{n-1} \frac{\prod_{i_1=0}^{4k+4} A_{2i_1} + u_0u_2u_4u_6 \sum_{i_2=0}^{4k+4} \left(B_{2i_2} \prod_{m=i_2+1}^{4k+4} A_{2m} \right)}{\prod_{i_1=0}^{4k+2} A_{2i_1} + u_0u_2u_4u_6 \sum_{i_2=0}^{4k+2} \left(B_{2i_2} \prod_{m=i_2+1}^{4k+2} A_{2m} \right)}$$

$$u_{8n+7} = u_7 \prod_{k=0}^{n-1} \frac{\prod_{i_1=0}^{4k+4} A_{2i_1+1} + u_1u_3u_5u_7 \sum_{i_2=0}^{4k+4} \left(B_{2i_2} \prod_{m=i_2+1}^{4k+4} A_{2m} \right)}{\prod_{i_1=0}^{4k+2} A_{2i_1+1} + u_1u_3u_5u_7 \sum_{i_2=0}^{4k+2} \left(B_{2i_2} \prod_{m=i_2+1}^{4k+2} A_{2m} \right)}$$

CONCLUSION

Lie symmetry is useful to solve nonlinear differential and difference equations. In this thesis, we review the symmetry method that can be used to solve the differential and difference equations. In particular, we use the symmetry method to give closed form of solutions of sixth order difference equation

$$u_{n+6} = \frac{u_n}{A_n + B_n u_n u_{n+3}} := \omega$$

where the initial values $u_0, u_1, \cdots, u_6$ are arbitrary nonzero real numbers and the eighth order difference equation

$$u_{n+8} = \frac{u_n}{A_n + B_n u_n u_{n+2} u_{n+4} u_{n+6}} := \omega$$

where the initial values $u_0, u_1, \cdots, u_8$ are arbitrary nonzero real numbers .

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